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本研究具有政策應用參考價值：■否 □是，建議提供機關
（勾選「是」者，請列舉建議可提供施政參考之業務主管機關）
本研究具影響公共利益之重大發現：□否 □是

中 華 民 國 110 年 10 月 31 日

中 文 摘 要 ： 令 $f(x) = x^{d} + c \in K[x]$ 其中 c 為參數，則 f(x) 是具有單一（有限）臨界點的多項式。我們探討 f(x) 的任意次迭代 f^{n}(x), 其中 n 為正整數所引起的迭代 Galois 擴張 K_{n}(f, \beta) = K(f^{-n}(\beta)), 其中 \beta 為 K中的元素及其迭代 Galois 群 G_{n}(\beta) = Gal(K_{n}(f, \beta)/K) 。我們考慮了三種情況：(1) K 是有限超越次數的函數體 (2) K 是一實二次體以及 (3) K 是一個配備一離散賦值及特徵為正的剩餘體的局部域。

中 文 關 鍵 詞 ： 樹棲表現論，迭代 Galois 擴張，迭代 Galois 群，有根樹，迭代圈積，單一臨界點，函數體，有限超越次數，實二次體，局部域。

英 文 摘 要 ： We study the iterated Galois group $G_{n}(\beta) = \gal(K_{n}(f, \beta)$ for $K_{n}(f, \beta) = K(f^{-n}(\beta))$ where $f(x)$ is the family of (finite) unicritical polynomials $f(x) = x^{d} + c \in K[x]$ with parameter $c $ ranges over a given field $K.$ Three cases are considered. Namely, (1) K is a function field of finite transcendence degree over $\bar{\QQ}$ (2) K is a real quadratic number field and (3) K is a complete local field with a discrete valuation with residue characteristic $p > 0.$

英 文 關 鍵 詞 ： Arboreal representation, iterated extension, iterated Galois group, rooted tree, iterated wreath product, unicritical polynomials, function field, finite transcendence degree, real quadratic field, complete local field.

一、中英文摘要與關鍵詞:

一、中文摘要: 令 $f(x) = x^d + c \in K[x]$ 其中 $c \in K$ 為參數，則 $f(x)$ 是具有單一（有限）臨界點的多項式。我們探討 $f(x)$ 的任意次迭代 $f^n(x), n \in \mathbb{N}$ 所引起的迭代 Galois 擴張 $K_n(f, \beta) = K(f^{-n}(\beta)), \beta \in K$ 及其迭代 Galois 群 $G_n(\beta) = \mathrm{Gal}(K_n(f, \beta)/K)$ 。我們考慮了三種情況：(1) K 是有限超越次數的函數體 (2) K 是一實二次體以及 (3) K 是一個配備一離散賦值及特徵為正的剩餘體的局部域。

關鍵詞: 樹棲表現論，迭代 Galois 擴張，迭代 Galois 群，有根樹，迭代圈積，單一臨界點，函數體，有限超越次數，實二次體，局部域。

Abstract. We study the iterated Galois group $G_n(\beta) = \mathrm{Gal}(K_n(f, \beta)$ for $K_n(f, \beta) = K(f^{-n}(\beta))$ where $f(x)$ is the family of (finite) unicritical polynomials $f(x) = x^d + c \in K[x]$ with parameter c ranges over a given field K. Three cases are considered. Namely, (1) K is a function field of finite transcendence degree over $\bar{\mathbb{Q}}$ (2) K is a real quadratic number field and (3) K is a complete local field with a discrete valuation with residue characteristic $p > 0$.

Key Words : Arboreal representation, iterated extension, iterated Galois group, rooted tree, iterated wreath product, unicritical polynomials, function field, finite transcendence degree, real quadratic field, complete local field.

二、前言：

In this two year project, our main focus is on the study of *iterated galois groups* which arises from iterations of some special classes of polynomials. We have reported partially on the progress of this project in the MoST report last year. This is an expansion of last year's report. During the project, we have published two joint papers [2, 3] and supervised two graduate students who got their master degrees in June, 2020. One of them is now a PhD student studying at Tsing-Hua University and another student now a research assistant at NCTS who is will study in Japan.

The question we study is related to the so-called *arboreal representation* which is initiated by Odoni [11]. To describe the problem, let K be a field, $f \in K(x)$ with $d = \deg f \geq 2$ and let $\beta \in \mathbb{P}^1(\bar{K})$. For $n \in \mathbb{N}$, let $K_n(f, \beta) = K(f^{-n}(\beta))$ be the field obtained by adjoining the nth preimages of β under f to $K(\beta)$ (by convention, we declare that $K(\infty) = K$). Set $K_\infty(f, \beta) = \bigcup_{n=1}^{\infty} K_n(f, \beta)$. For $n \in \mathbb{N} \cup \{\infty\}$, define $G_n(f, \beta) = \mathrm{Gal}(K_n(f, \beta)/K(\beta))$. We will write $G_n(\beta)$ and $K_n(\beta)$, suppressing the dependence on f if there is no ambiguity. Put $G_\infty(\beta) = \mathrm{Gal}(K_\infty(f, \beta)/K)$ which is the inverse limit of $G_n(f, \beta)$. The set $\cup_{n=1}^{\infty} f^{-n}(\beta)$ of preimages of β under f has a natural structure of rooted tree. It follows from Galois theory, the action of the Galois group $G_\infty(\beta)$ acting on the infinite set $\cup_{n=1}^{\infty} f^{-n}(\beta)$ gives an embedding of $G_\infty(\beta)$ into $\mathrm{Aut}(T_\infty^d)$, the automorphism group of an infinite d-ary rooted tree T_∞^d. The question we're concerned with is, under this *tree representation* (arboreal Galois representation), how large the image of the group $G_\infty(\beta)$ is as a subgroup of the automorphsim group $\mathrm{Aut}(T_\infty^d)$ of T_∞^d?

In the two papers [2, 3] and [9, 7] , we consider the family of unicritical polynomials, i.e. one-variable polynomials $f(x)$ of degree $d \geq 2$ over a field K with only one critical point. Up to change of variables, such polynomials can be expressed as $f(x) = x^d + c$ with parameter $c \in K$, which represents all polynomials defined over K with precisely one (finite) critical point. There are several cases that we've considered: (1) K is a function field of finite transcendence degrees over the field of rational numbers $\mathbb{Q}$, (2) K is a quadratic number field and (3) K is a finite extension of the field of p-adic numbers.

三、研究目的：

There has been much work on the problem of determining whether or not the index $[\mathrm{Aut}(T_\infty^d) : G_\infty(\beta)]$ is finite. The group $G_\infty(\beta)$ is the image of an arboreal Galois representation, so this finite index problem is a natural analog in arithmetic dynamics of the finite index problem for the ℓ-adic Galois representations associated to elliptic curves, resolved by Serre's celebrated Open Image Theorem [12]. By work of Odoni [11], one expects that a generically chosen rational function has a surjective arboreal representation, i.e., that $[\mathrm{Aut}(T_\infty^d) : G_\infty(\beta)] = 1$.

If the field K contains a primitive dth root of unity, then it is easy to show that for the family of unicritical polynomials f, $G_\infty(\beta)$ sits in $[C_d]^\infty$, the infinite iterated wreath product of the cyclic group C_d (with d elements). As $\mathrm{Aut}(T_n^d) \cong [S_d]^n$, unless $d = 2$ in which case $S_2 = C_2$, we have $[\mathrm{Aut}(T_\infty^d) :$

$[C_d]^\infty] = \infty$ if $d \geq 3$. Thus it is impossible for $G_\infty(\beta)$ to have finite index within this family (except when $d = 2$). However, this simply means that, given the constraint on the size of $G_\infty(\beta)$, we should ask a different finite index question. In the following, we divide our discussion by considering (1) K is the function field of a smooth irreducible curve defined over $\bar{\mathbb{Q}}$, (2) K is a function field of finite transcendence degrees over the field of rational numbers $\mathbb{Q}$, (3) K is a quadratic number field and (4) K is a finite extension of the field of p-adic numbers.

(1). In the paper [2], we considered the case where K is the function field of a smooth irreducible curve defined over $\bar{\mathbb{Q}}$. A necessary and sufficient conditions are given to guaranteed that $G_\infty(\beta)$ has finite index in $[C_d]^\infty$ when base field K is of transcendence degree one over $\mathbb{Q}$. We also established a result concerning the notion of *eventually stable* which is crucial in proving the finite index of $G_\infty(\beta)$ as a subgroup inside $[C_d]^\infty$ for the family of unicritical polynomials.

(2) It is natural to ask if similar results could hold for the case where K has higher transcendence degree over $\mathbb{Q}$. This question is answered in the paper [3]. In fact, in this paper we aim at exploring a more general question stated as follow.

Question A *Let K be a number field or a function field of finite transcendence degree over a field of characteristic 0, let X be a smooth projective variety defined over K and let $\Phi : X \longrightarrow X$ be a polarizable endomorphism, meaning that there exists an ample line bundle $\mathcal{L}$ on X such that $\Phi^*\mathcal{L}$ is linearly equivalent to $\mathcal{L}^{\otimes d}$ for some integer $d > 1$.*

For each positive integer n, we have that Φ^n induces an inclusion of the function field $K(X)$ into itself; we let $G_n(\Phi, X)$ be the Galois group of the Galois closure of $K(X)$ over itself with respect to this inclusion. We let $G_\infty := G_\infty(\Phi, X)$ be the inverse limit of these groups $G_n(\Phi, X)$.

For each point $x \in X(K)$, we let $G_n(\Phi, x)$ be the Galois group of $K(\Phi^{-n}(x))$ over $K(x)$. We let $G_\infty(x) := G_\infty(\Phi, x)$ be the inverse limit of the groups $G_n(\Phi, x)$. We have that there is a natural embedding of $G_\infty(x)$ inside G_∞.

Then is it true that at least one of the following statements must hold?

(A) The index $[G_\infty(\Phi, X) : G_\infty(\Phi, x)]$ is finite.

(B) The point x lies in the strict forward orbit of a point in the ramification locus of Φ.

(C) The point x lies on a proper subvariety $Y \subset X$ that is invariant under a non-identity self-map $\Psi : X \longrightarrow X$ with the property that $\Phi^n \circ \Psi = \Psi \circ \Phi^n$ for some some positive integer n.

Jones [8, Conjecture 3.11] has proposed a similar conjecture for quadratic rational functions $\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ over number fields. Here we ask a more general question which has not previously been posed for arbitrary function fields K (of characteristic 0) or for self-maps of higher dimensional varieties. In this paper we discuss some special cases of Question A which generally fall outside the scope of the previous studies of the arboreal Galois representation associated to a dynamical system. In particular, we treat the case when K is a function field of transcendence degree greater than one (and $\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is a polynomial mapping), and also we discuss several cases when X is a higher dimensional variety.

(3). In the case where K is a number field, one also expects that $G_\infty(\beta)$ has finite index in $[C_d]^\infty$ for general values of $c \in K$. However, it can be challenging to specify conditions on c so that $G_\infty(\beta)$ is isomorphic to $[C_d]^\infty$. In [13], M. Stoll studied this question for the case where $\beta = 0$ and c varies over the set of integers. Several non-trivial sufficient conditions are established by the author. It is natural to extend Stoll's criterion to a more general situations. Motivated by Stoll's result, we considered the case where K is a real quadratic number field and c varies among the ring of integers of K. Part of this project is carried out in [9] where we follow Odoni as well as Stoll's ideas to study the quadratic family $f(x) = x^2 + c$ under the condition that quadratic number field K whose class number is one. Several sufficient conditions for the parameter c such that $G_\infty(\beta)$ is isomorphic to $[C_d]^\infty$ are obtained.

(4). It also makes sense to study the same quesiton in the case where K is a finite extension field of the field $\mathbb{Q}_p$ of p-adic numbers. One expects that the Galois group $G_\infty(\beta)$ is much smaller than expected since $\mathrm{Gal}(\bar{K}/K)$ is a solvable group where $\bar{K}$ denotes an algebraic closure of K. Generalizing the results in [1], we explored in [1] that indeed the index $[[C_d]^\infty : G_\infty(\beta)]$ is infinite for the family of polynomials $f(x) = x^d + c,\ c \in K$ with some restrictions on the degree d.

四、研究方法、結果與討論：

(1). Before stating the main result in [2], we set some notation. The field K refers to a function field of transcendence degree 1 over its field of constants $\bar{\mathbb{Q}}$. In other words, K is the function field of a smooth, projective, irreducible curve C over $\bar{\mathbb{Q}}$. We say $\beta \in \bar{K}$ is periodic for f if $f^n(\beta) = \beta$ for some $n \geq 1$, and we say β is preperiodic for f if $f^m(\beta)$ is periodic for some $m \geq 0$. Finally, we say that β is postcritical for f if $f^n(\alpha) = \beta$ for some $n \geq 1$ and some critical point α of f.

With this notation, the first main result of [2] is the following.

Theorem 1 *Let $q = p^r$ $(r \geq 1)$ be a power of the prime number p, let $c \in K \setminus \bar{\mathbb{Q}}$, let $f(x) = x^q + c \in K[x]$ and let $\beta \in K$. Then the following are equivalent:*

1. *The point β is neither periodic nor postcritical for f.*
2. *The group $G_\infty(\beta)$ has finite index in $[C_q]^\infty$.*

We note that in the case where $q = 2$, this means that $G_\infty(\beta)$ has finite index in $\mathrm{Aut}(T^2_\infty)$. For larger q this index is infinite, as mentioned previously.

Remark 2 *In general one needs to rule out postcritically finite (PCF) maps in order to obtain a finite index result, as in the main result of [5]. The reason we do not need to do this in Theorem 1 is that a PCF polynomial of the form $f(x) = x^q + c$ is automatically isotrivial meaning that there exists a linear* *change of variable such that the parameter* $c \in \bar{\mathbb{Q}}$ *. For PCF maps, because c satisfies an equation of the form* $f^n(c) = f^m(c)$ *for some* $n > m \geq 0$*, and so* $c \in \bar{\mathbb{Q}}$*. For isotrivial polynomials the PCF distinction regains its importance and it's treated separately in the paper.*

One of the key steps in the proof of Theorem 1 is an eventual stability result. As is usual in arithmetic dynamics, we say that the pair (f, β) is eventually stable over the field K if the number of irreducible K-factors of $f^n(x) - \beta$ is uniformly bounded for all n. The following result extends [2] to function fields of arbitrary transcendence degree.

Theorem 3 *Let* $q = p^r$ $(r \geq 1)$ *be a power of the prime number* p*. Let* $f \in K[x]$ *be a polynomial of the form* $x^q + t$ *where* $t \notin \bar{\mathbb{Q}}$*. Then for any non-periodic* $\beta \in K$*, the pair* (f, β) *is eventually stable over* K*.*

All the methods used in the proof of Theorem 1 work for unicritical polynomials of any degree d, except that we need the degree to be a prime power for proving the eventual stability of (f, β). This is the reason that Theorem 1 is stated only for the case where d is a power of a prime number.

We also prove the following disjointness theorem for fields generated by inverse images of different points under different maps.

Theorem 4 *Let* K *be the function field of a smooth, projective variety defined over* $\bar{\mathbb{Q}}$*. For* $i = 1, \ldots, n$ *let* $f_i(x) = x^q + c_i \in K[x]$*, where* $c_i \notin \bar{\mathbb{Q}}$*, and let* $\alpha_i \in K$*. Suppose that there are no distinct* i, j *with the property that* (α_i, α_j) *lies on a curve in* $\bar{A}^2$ *that is periodic under the action of* $(x, y) \mapsto (f_i(x), f_j(y))$*. For each* i*, let* M_i *denote* $K_\infty(f_i, \alpha_i)$*. Then for each* $i = 1, \ldots, n$*, we have that*

$$\left[M_i \cap \left(\prod_{j \neq i} M_j \right) : K \right] < \infty.$$

Theorem 4 also has a natural interpretation as a finite index result across pre-image trees of several points

Remark 5 *In Theorem 4, since each* c_i *is not in* $\bar{\mathbb{Q}}$*, then [**?**, Theorems 1.4 and 1.5] yield the following:*

- *if there is no* $(d-1)$*-st root of unity* ζ *such that* $c_i = \zeta c_j$*, then* (α_i, α_j) *lies on a curve in* $\bar{A}^2$ *periodic under the action of* $(x, y) \mapsto (f_i(x), f_j(y))$ *if and only if either* α_i *is periodic for* f_i *or* α_j *is periodic for* f_j*.*

- *if there is some* $(d-1)$*-st root of unity* ζ *such that* $c_i = \zeta c_j$*, then* (α_i, α_j) *lies on a periodic curve under the action of* $(x, y) \mapsto (f_i(x), f_j(y))$ *if and only if either* α_i *is periodic for* f_i*, or* α_j *is periodic for* f_j*, or* $x = \zeta f_j^m(y)$ *for some* $m \geq 0$*, or* $y = f_j^m(\zeta^{-1}x)$ *for some* $m \geq 1$*.*

Remark 6 *In light of Odoni's work, unicritical polynomials with degree* $d \geq 3$ *cannot be considered generic from the point of view of arboreal Galois theory (indeed, they are not a generic family in the moduli space of degree* d *polynomials in any reasonable sense). There are other families of polynomials and rational functions (such as postcritically finite maps) that arise as* *obstructions to any potential classification of finite index arboreal representations – see [8, Section 3] and [5, Prop 3.3] for examples. One might hope that in these "exceptional" families, something similar to Theorem 1 could hold, in that a broad finite index result could be established for a natural overgroup other than* $\mathrm{Aut}(T^d_\infty)$. *We hope to explore this in future work.*

(2). In paper [3], we treat Question A for the special case where $X = \mathbb{P}^1$ and Φ is a unicritical polynomial (which we denote by f), while K is the function field of an arbitrary smooth projective variety defined over $\mathbb{Q}$. The following result is an extension of Theorem 1 to function fields of arbitrary finite transcendence degree and it represents a special case of our Question A.

Theorem 7 *Let K be the function field of a smooth projective variety defined over $\bar{\mathbb{Q}}$. Let $q = p^r$ $(r \geq 1)$ be a power of the prime number p, let $c \in K \smallsetminus \bar{\mathbb{Q}}$, let $f(x) = x^q + c \in K[x]$ and let $\beta \in K$. Then the following are equivalent:*

1. *The point β is neither periodic nor postcritical for f.*
2. *The group $G_\infty(\beta)$ has finite index in G_∞.*

We also extend the eventually stablility result Theorem 3 to the case of function fields of arbitrary transcendence degree.

Theorem 8 *Let K be the function field of a smooth projective variety defined over $\bar{\mathbb{Q}}$. Let $q = p^r$ $(r \geq 1)$ be a power of the prime number p. Let $f \in K[x]$ be a polynomial of the form $x^q + c$ where $c \notin \bar{\mathbb{Q}}$. Then for any non-periodic $\beta \in K$, the pair (f, β) is eventually stable over K.*

We also prove the following disjointness theorem for fields generated by inverse images of different points under different maps; our result is a generalization of Theorem 4

Theorem 9 *Let K be the function field of a smooth projective variety defined over $\bar{\mathbb{Q}}$. For $i = 1, \ldots, m$ let $f_i(x) = x^q + c_i \in K[x]$, where $c_i \notin \bar{\mathbb{Q}}$, and let $\alpha_i \in K$. Suppose that there are no distinct i, j with the property that (α_i, α_j) lies on a curve in $\bar{A}^2$ that is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$. For each i, let M_i denote $K_\infty(f_i, \alpha_i)$. Then for each $i = 1, \ldots, n$, we have that*

$$\left[M_i \cap \left(\prod_{j \neq i} M_j \right) : K \right] < \infty.$$

Theorem 9 coupled with Theorem 7 yields the following special case of Question A.

Theorem 10 *Let K be the function field of a smooth projective variety defined over $\bar{\mathbb{Q}}$, let q be a power of the prime number p, and let m be a positive integer. For $i = 1, \ldots, m$ let $f_i(x) = x^q + c_i \in K[x]$, where $c_i \notin \bar{\mathbb{Q}}$, and let $\alpha_i \in K$. We let $\underline{\alpha} := (\alpha_1, \ldots, \alpha_m)$ and let $\Phi := (f_1, \ldots, f_m)$ acting on $X := (\mathbb{P}^1)^m$. Then let $G_n(\Phi, \underline{\alpha})$ be the Galois group of $K(\Phi^{-n}(\underline{\alpha}))$ over K. We let $G_\infty(\underline{\alpha}) := G_\infty(\Phi, \underline{\alpha})$ be the inverse limit of the groups $G_n(\Phi, \underline{\alpha})$.*

Then at least one of the following must hold:

(A) $[G_\infty(\Phi, X) : G_\infty(\Phi, \underline{\alpha})]$ *is finite;*

(B) $\underline{\alpha}$ *lies in the strict forward orbit of a point in the ramification locus of* Φ*; or*

(C) $\underline{\alpha}$ *lies on a proper subvariety* $Y \subset X$ *that is invariant under a non-identity self-map* $\Psi : X \longrightarrow X$ *with the property that* $\Phi \circ \Psi = \Psi \circ \Phi$.

Finally, using a similar strategy as employed in [5], we obtain the following result regarding the arboreal Galois representation associated to cubic polynomials. Once again, we use the notation from Question A for G_∞ and $G_\infty(x)$ and since both groups lie naturally inside $\mathrm{Aut}(T^3_\infty)$, then in order to prove the finiteness of the index of $G_\infty(x)$ inside G_∞, it suffices to prove $[\mathrm{Aut}(T^3_\infty) : G_\infty(x)] < \infty$.

Theorem 11 *Let* K *be the function field of a smooth projective variety defined over* $\bar{\mathbb{Q}}$. *Let* $f \in K[x]$ *be a cubic polynomial. Assume that* f *is not isotrivial over* $\bar{\mathbb{Q}}$, *that* β *is not periodic or postcritical for* f, *that the pair* (f, β) *is eventually stable, and that* f *has distinct finite critical points* γ_1, *and* γ_2, *and* $f^n(\gamma_1) \neq f^n(\gamma_2)$ *for all* $n \geq 1$. *Then*

$$[\mathrm{Aut}(T^3_\infty) : G_\infty(\beta)] < \infty.$$

Our proofs will rely mainly on specialization techniques in order to extend the results of [2] to the generality from the current situation.

(3). Instead of taking K to be a function field over $\bar{\mathbb{Q}}$, one can consider the specialization of the parameter c to an algebraic number. In [13], the author considers the case where the degree $d = 2$ and c ranges over the set of integers while β is taken to be 0, the unique critical point of the polynomial $f(x) = x^2 + c$ in question. Sufficient conditions for parameters $c \in \mathbb{Z}$ are established so that $G_\infty(0) \simeq [C_2]^\infty$. In our situation, we consider K be a real quadratic number field whose ideal class number $h_K = 1$. As in [13], we also give some sufficient conditions such that $G_\infty(0) \simeq [C_2]^\infty$. Specifically, we have the following.

Theorem 12 *Let* $f(X) = X^2 + c \in K[X]$, *then* $G_\infty(0) \simeq [C_2]^\infty$ *for the following three cases:*

(1) $K = \mathbb{Q}(\sqrt{2})$ *and* $c = a + b\sqrt{2}$ *where* a, b *are positive odd integers.*

(2) $K = \mathbb{Q}(\sqrt{2p})$ *where* p *is a prime number and* $h_K = 1$ *such that* $p \equiv 3 \pmod{()4}$ *and* $c = a + b\sqrt{2p}$ *where* a, b *are positive integers satisfying* $a \equiv 1 \pmod{2p}$ *and* $b \equiv 1 \pmod{2}$.

(3) $K = \mathbb{Q}(\sqrt{p})$ *where* p *is a prime number and* $h_K = 1$ *such that* $p \equiv 1 \pmod{()4}$ *and* $c = a + b\left(\frac{1+\sqrt{p}}{2}\right)$ *where* a, b *are positive integers satisfying* $a \equiv -1 \pmod{4p}$ *and* $b \equiv 2p \pmod{4p}$

The strategies for proving Theorem 12 mainly follow from the ideas in [13]. A key ingredient is Kummer Theory.

(3). Motivated by the paper [1], we consider K be a finite extension of the field of p-adic numbers in the third part. More precisely, the authors investigate the case where K is a local field completed with respect to a discrete valuation v of residue characteristic $p > 0$. For the polynomial $f(x) = x^d + c$ with $c \in K$, examine the Galois group $G_n(\beta)$ and ramification groups of the extension $K_n(f, \beta)$ over K for $\beta \in K$. They describe the dependence of the behavior on $v(c)$ as well as conditions on d: (i) $p \nmid d$ and (ii) $d = p$. As the results are not complete for the degrees of $f(x)$, we explore further to extend the study to the general cases where the degree $d = p^r$ with $r \geq 1$. We also give a classification of the behavior of the ramifications of $K_\infty(\beta)/K$ according to the valuations $v(c)$ and $v(\beta)$. The results are collected as follows.

Theorem 13 *Let K be a finite extension over $\mathbb{Q}_p$, and let $f(X) = X^{p^r} - c$.*

1. *If $v(c) \geq -p$, then $K_\infty(\beta)/K$ is infinitely ramified.*
2. *If $v(c) < \frac{-p}{p-1} - \frac{p^r}{p^r-1}(r-1)$, then $K_\infty(\beta)/K$ is a finite extension.*

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108年度專題研究計畫成果彙整表

<table>
<tr><td colspan="3">計畫主持人：夏良忠</td><td colspan="3">計畫編號：108-2115-M-003-005-MY2</td></tr>
<tr><td colspan="6">計畫名稱：帶參數海價映射族群正則高度函數變異之研究</td></tr>
<tr><td colspan="3">成果項目</td><td>量化</td><td>單位</td><td>質化
（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）</td></tr>
<tr><td rowspan="6">國內</td><td rowspan="6">學術性論文</td><td>期刊論文</td><td>0</td><td rowspan="2">篇</td><td></td></tr>
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<tr><td>專書</td><td>0</td><td>本</td><td></td></tr>
<tr><td>專書論文</td><td>0</td><td>章</td><td></td></tr>
<tr><td>技術報告</td><td>0</td><td>篇</td><td></td></tr>
<tr><td>其他</td><td>0</td><td>篇</td><td></td></tr>
<tr><td rowspan="6">國外</td><td rowspan="6">學術性論文</td><td>期刊論文</td><td>2</td><td rowspan="2">篇</td><td>(1) A. Bridy , J. Doyle, D. Ghioa, L.-C. Hsia, AND T. Tucker, Finite index theorem for iterated. Galois groups of unicritical polynomials, Trans. AMS, vol 374, no. 1, (2021), 733-752.
(2) A. Bridy , J. Doyle, D. Ghioa, L.-C. Hsia, AND T. Tucker, A question for iterated Galois groups in arithmetic dynamics, Canadian Mathematical Bulletin , Volume 64 , Issue 2 , June 2021 , pp. 401 - 417.</td></tr>
<tr><td>研討會論文</td><td>0</td><td></td></tr>
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<tr><td rowspan="10">參與計畫人力</td><td rowspan="5">本國籍</td><td>大專生</td><td>5</td><td rowspan="10">人次</td><td>學習本計畫相關知識。</td></tr>
<tr><td>碩士生</td><td>3</td><td>三名碩士生，於 109 學年度第二學期畢業。</td></tr>
<tr><td>博士生</td><td>0</td><td></td></tr>
<tr><td>博士級研究人員</td><td>0</td><td></td></tr>
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<tr><td>專任人員</td><td>0</td><td></td></tr>
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其他成果 （無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）	