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維騰拉普拉斯算子和擬埃爾米特瑞奇曲率

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中 文 摘 要 ： 根據Munteanu-Wang 的工作，我們想延伸他們的結果到擬埃爾米特2n+1維度的流形。首先，根據在黎曼流形和在CR流形上的波切內爾公式的不同形式，在CR流形上處理分析問題最主要問題是處理T方向的微分。所以我首先的工作是得到T方向微分是零的結果，則由此我們可以建構特徵值的估計。此外，我們也得到緊緻擬埃爾米特流形上根據Kohn算子的特徵值得到Paneitz算子的特徵值估計。

中 文 關 鍵 詞 ： 擬埃爾米特流形, 次拉普拉斯算子, 波切內爾公式

英 文 摘 要 ： From the work of O. Munteanu and J. Wang, we extend their some results to the pseudohermitian (2n+1)-manifold. In fact, according to the Bochner formula in CR manifold, the derivative along to the direction T is the key point whenever we consider the analytic work on CR manifold. Hence, our first work is the vanishing property of T-derivative, and then we have the estimate of eigenvalue on CR manifold.

英 文 關 鍵 詞 ： pseudohermitian manifold, sub-Laplacian, Bochner formula

報告內容:

(一). 前言、研究目的:

From the work of O. Munteanu and J. Wang, we extend their some results to the pseudohermitian $(2n+1)$-manifold. In fact, according to the Bochner formula in CR manifold, the derivative along to the direction T is the key point whenever we consider the analytic work on CR manifold. Hence, our first work is the vanishing property of T-derivative, and then we have the estimate of eigenvalue on CR manifold.

(二). 文獻探討及研究方法:

In 1976, R.M. Schoen and S.T. Yau ([SY]) proved the Liouville type properties of harmonic map from complete manifolds with nonnegative Ricci curvature to compact manifolds with nonpositive curvature. In 2001, P. Li and J. Wang ([LW]) showed the Liouville type properties and splitting type properties of harmonic function on complete manifold with positive spectrum λ and the Ricci curvature has a lower bound depending on λ. In 2006, P. Li and J. Wang ([LW2]) extended their results to complete manifolds with the condition (P_ρ). Recently, S.C. Chang and J.T. Chen and S.W. Wei ([CCW]) considered the p-harmonic function on complete manifolds with (P_ρ) and the Ricci curvature has a lower bound depending on ρ, then the Liouville type properties are still valid on these manifolds. Recently, In 2011, O. Munteanu and J.-P. Wang ([MW1] [MW2]) extend the Liouville type properties and splitting type properties to smooth metric measure spaces.

For a complete pseudohermitian $(2n+1)$-manifold $(M, J, \theta, d\mu)$ with $d\mu = e^{-\phi(x)}\theta \wedge (d\theta)^n$, we define the weighted sub-Laplacian L by

$$Lu(x) \equiv \Delta_b u(x) - \nabla_b \phi(x) \cdot \nabla_b u(x)$$

which is the Euler-Lagrangian equation of the Dirichlet energy

$$E(f) = \int_M |\nabla_b f|^2 \, d\mu$$

for $f \in C_0^\infty(M)$.

Definition 1 *A function u is called ϕ-pseudoharmonic function iff $Lu = 0$.*

We also let the ∞-dimensional Bakry-Emery pseudohermitian Ricci curvature $Ric(L)$ by

$$Ric(L)(W, W) \equiv R_{\alpha\overline{\beta}} W_\alpha W_{\overline{\beta}} + \mathrm{Re}\left[\phi_{\alpha\overline{\beta}} W_{\overline{\alpha}} W_\beta\right]$$

and the torsion $Tor(L)$ by

$$Tor(L) \equiv 2\,\mathrm{Re}\left(\left(i(n-2) A_{\overline{\alpha\beta}} - \phi_{\overline{\alpha\beta}}\right) W_\alpha W_\beta\right)$$

for $W = W^{\alpha} Z_{\alpha} \in T_{1,0}$.

Recall the CR Bochner formula for Δ_b ([Gr]): Assume u is a real smooth function on (M, J, θ). Then

$$\frac{1}{2}\Delta_b\left(|\nabla_b u|^2\right) = \left|\left(\nabla^H\right)^2 u\right|^2 + \langle\nabla_b u, \nabla_b \Delta_b u\rangle + 2\langle J\nabla_b u, \nabla_b u_0\rangle + \left[2Ric_M - (n-2)\,Tor\right]\left((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}\right). \tag{1}$$

We also recall the following Kato's inequality ([CtCK]): if u is a C^2 function on M, then

$$\left|\left(\nabla^H\right)^2 u\right|^2 \geq \frac{n}{2}u_0^2 + \frac{1}{2n}\left(\Delta_b u\right)^2 \quad \text{and} \quad u^2\left|\left(\nabla^H\right)^2 u\right|^2 \geq \frac{\left|\nabla_b h^2\right|^2}{4} \tag{2}$$

for all $x \in M$. Here $h = |\nabla_b u|$.

Lemma 2 *Let $(M, J, \theta, d\mu)$ be a complete pseudohermitian $(2n+1)$-manifold. For a real smooth function u on M,*

$$\frac{1}{2}L|\nabla_b u|^2 = \left|\left(\nabla^H\right)^2 u\right|^2 + \langle\nabla_b u, \nabla_b L u\rangle + 2\langle J\nabla_b u, \nabla_b u_0\rangle + \left[2Ric(L) - Tor(L)\right]\left((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}\right) \tag{3}$$

(三). 結果與討論:

Vanishing of $\mathbf{u}_0$:

Theorem 3 *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold of infinite ϕ-volume. Suppose that*

$$Ric\,(L)\,(Z, Z) - Tor\,(L)\,(Z, Z) \geq -K\,|Z|^2$$

for all $Z \in T_{1,0} \oplus T_{0,1}$ and some constant $k \geq 0$. If u is a ϕ-pseudoharmonic function with finite energy $\int_M |\nabla_b u|^2 e^{-\phi} d\mu < \infty$ and $[L, T]u = 0$. Then $u_0 = 0$.

Let $\eta : M \longrightarrow [0, 1]$ be a cut off function such that

$$\eta = \begin{cases} 1 & \text{on } B\,(R)\,, \\ 0 & \text{on } M \backslash B\,(2R)\,, \end{cases}$$

$\eta \leq 1$ and

$$|\nabla_b \eta\,(x)|^2 \leq c\eta R^{-2} \text{ on } B\,(2R) \backslash \overline{B\,(R)}.$$

By using (1), Kato's inequality and integration by parts, we have the following lemmas.

Lemma 4 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold. Assume u is a ϕ-pseudoharmonic function with $[L, T]\,u = 0$ on M. Then for $h = |\nabla_b u|$ and any $0 < \varepsilon_0,\ \varepsilon_1 < 1/2$, $a > 0$,*

$$\begin{aligned} &(1 - \varepsilon_0 - \varepsilon_1) \int_M \eta^2 \left|\nabla_b h\right|^2 d\mu + \left(\tfrac{\varepsilon_0 n}{2} - \tfrac{c_1 a}{R^2}\right) \int_M \eta^2 u_0^2 d\mu \\ &+ \left(a - \tfrac{1}{\delta_1}\right) \int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu \\ &+ \int_M \eta^2 \left(\left[Ric\,(L) - Tor\,(L)\right]\right)(\nabla_b u, \nabla_b u)\, d\mu \\ &\leq \left(\tfrac{c_2}{\varepsilon_1 R^2} + \delta_1\right) \int_M \eta h^2 d\mu, \end{aligned} \tag{4}$$

whenever R large enough.

Proof. Using (2), we have

$$\begin{aligned} \left|\left(\nabla^H\right)^2 u\right|^2 &= (1 - \varepsilon_0) \left|\left(\nabla^H\right)^2 u\right|^2 + \varepsilon_0 \left|\left(\nabla^H\right)^2 u\right|^2 \\ &\geq (1 - \varepsilon_0)\,|\nabla_b h|^2 + \varepsilon_0 \left|\left(\nabla^H\right)^2 u\right|^2 \end{aligned}$$

where $0 < \varepsilon_0 < 1/2$. If we multiply both sides of (3) by η and integrate over M, then one has

$$\begin{aligned}\frac{1}{2}\int_M \eta^2 L\left(h^2\right) d\mu \quad &\geq \quad (1-\varepsilon_0)\int_M \eta^2 \left|\nabla_b h\right|^2 d\mu + \int_M \eta^2 \Bigg(\varepsilon_0 \left| \left(\nabla^H\right)^2 u \right|^2 \\ &\quad + \left[Ric\left(L\right) - Tor\left(L\right)\right]\left(\nabla_b u, \nabla_b u\right) \\ &\quad + 2\left\langle J\nabla_b u, \nabla_b u_0 \right\rangle \Bigg) d\mu, \end{aligned} \tag{5}$$

On the other hand, by Yang's inequality

$$\begin{aligned}\frac{1}{2}\int_M \eta^2 L\left(h^2\right) d\mu \quad &= \quad \frac{-1}{2}\int_M \left\langle \nabla_b \eta^2, \nabla_b h^2 \right\rangle d\mu \\ &\leq \quad \varepsilon_1 \int_M \eta^2 \left|\nabla_b h\right|^2 d\mu + \frac{1}{\varepsilon_1}\int_M \left|\nabla_b \eta\right|^2 h^2 d\mu,\end{aligned}$$

where $0 < \varepsilon_1 < 1/2$. Hence (5) implies

$$\begin{aligned}&(1-\varepsilon_0-\varepsilon_1)\int_M \eta^2 \left|\nabla_b h\right|^2 d\mu + \int_M \eta^2 \Bigg(\varepsilon_0 \left| \left(\nabla^H\right)^2 u \right|^2 \\ &+ \left[Ric\left(L\right) - Tor\left(L\right)\right]\left(\nabla_b u, \nabla_b u\right) + 2\left\langle J\nabla_b u, \nabla_b u_0 \right\rangle \Bigg) d\mu \\ &\leq \frac{1}{\varepsilon_1}\int_M \left|\nabla_b \eta\right|^2 h^2 d\mu.\end{aligned} \tag{6}$$

Now we claim that

$$\int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu \quad \leq \quad 9\int_M \eta \left|\nabla_b \eta\right|^2 u_0^2 d\mu. \tag{7}$$

By using $\left[L, T\right] u = 0$, we have $L\left(u_0\right) = \left(Lu\right)_0 = 0$, then

$$\frac{1}{2}L\left(u_0^2\right) \quad = \quad \left|\nabla_b u_0\right|^2. \tag{8}$$

We multiply both sides of (8) by η^3 and integrate over M,

$$\frac{1}{2}\int_M \eta^3 L\left(u_0^2\right) d\mu = \int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu. \tag{9}$$

Due to integration by parts,

$$\begin{aligned}\frac{1}{2}\int_M \eta^3 L\left(u_0^2\right) d\mu \quad &= \quad = -3\int_M \eta^2 \left\langle \nabla_b \eta, u_0 \nabla_b u_0 \right\rangle d\mu \\ &\leq \quad 3\int_M \eta^2 \left|\nabla_b \eta\right| \left|u_0\right| \left|\nabla_b u_0\right| d\mu \\ &\leq \quad \frac{1}{2}\int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu + \frac{9}{2}\int_M \eta \left|\nabla_b \eta\right|^2 u_0^2 d\mu.\end{aligned} \tag{10}$$

Then (7) follows by (9) and (10).

Now combining 6 and 7, we obtain

$$\begin{aligned}&(1-\varepsilon_0-\varepsilon_1)\int_M \eta^2 \left|\nabla_b h\right|^2 d\mu + D \\ &+ \int_M \eta^2 \left[Ric\left(L\right) - Tor\left(L\right)\right]\left(\nabla_b u, \nabla_b u\right) d\mu \\ \leq \quad &\frac{1}{\varepsilon_1}\int_M \left|\nabla_b \eta\right|^2 h^2 d\mu.\end{aligned} \tag{11}$$

Here

$$D = \int_M \eta^2 \left(\varepsilon_0 \left| \left(\nabla^H\right)^2 u \right|^2 + 2 \langle J\nabla_b u, \nabla_b u_0 \rangle \right) d\mu + a \int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu$$
$$-9a \int_M \eta \left|\nabla_b \eta\right|^2 u_0^2 d\mu,$$

and a is a constant will be determined later.

By (2) and

$$2 \int_M \eta^2 h \left|\nabla_b u_0\right| d\mu \le \delta_1 \int_M \eta h^2 d\mu + \tfrac{1}{\delta_1} \int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu,$$

we have

$$\begin{aligned} D \ \ge\ & \left(\tfrac{\varepsilon_0 n}{2} - \tfrac{c_1 a}{R^2}\right) \int_M \eta^2 \left|u_0\right|^2 d\mu - \delta_1 \int_M \eta h^2 d\mu \\ & + \left(a - \tfrac{1}{\delta_1}\right) \int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu \end{aligned}$$

since $\left|\nabla_b \eta\right|^2 \le \frac{c\eta}{R^2}$. Hence Lemma follows. ■

Proof of Theorem 3:

Proof. Suppose $h \in L^2(M)$ and $Ric(L) - Tor(L) \ge -K$ for some $K \ge 0$ is a constant. Then (4) becomes

$$\begin{aligned} & (1 - \varepsilon_0 - \varepsilon_1) \int_M \eta^2 \left|\nabla_b h\right|^2 d\mu + \left(\tfrac{\varepsilon_0 n}{2} - \tfrac{c_1 a}{R^2}\right) \int_M \eta^2 u_0^2 d\mu \\ & + \left(a - \tfrac{1}{\delta_1}\right) \int_M \eta^3 \left|\nabla_b u_0\right|^2 d\mu \\ \le\ & \left(\tfrac{c_2}{\varepsilon_1 R^2} + \delta_1 + K\right) \int_M \eta h^2 d\mu, \end{aligned}$$

here $0 < \varepsilon_0,\ \varepsilon_1 < 1/2$. Thus, if we select $a = \varepsilon_0 R$ and $\delta_1 = 2a^{-1}$, then

$$\tfrac{\varepsilon_0 n}{4} \int_{B(R)} u_0^2 d\mu + \tfrac{\varepsilon_0 R}{2} \int_{B(R)} \left|\nabla_b u_0\right|^2 d\mu \ \le\ \left(\tfrac{c}{\varepsilon_1 R^2} + \tfrac{2}{\varepsilon_0 R} + K\right) \int_{B(2R)} h^2 d\mu, \tag{12}$$

whenever R large enough.

Notice that $\int_M h^2 d\mu < \infty$. Now we select $\varepsilon_0 = R^{-1/2}$ and let $R \to \infty$, then we obtain $\int_M \left|\nabla_b u_0\right|^2 d\mu = 0$. This means that u_0 must be a constant on M.

Besides, if we select $\varepsilon_0 = \frac{1}{4}$ and $R \to \infty$, (12) also implies $\frac{n}{16} \int_M u_0^2 d\mu \le K \int_M h^2 d\mu < \infty$, which is $u_0^2 \cdot vol_\phi(M) < \infty$. Then u_0 must be zero if the ϕ-volume of M is infinite. ■

Now we define that u is a ϕ-harmonic function on (M, g_ε), where g_ε is the Webster (adapted) metric of M, if

$$0 = L_\varepsilon u := \Delta_\varepsilon u - \nabla_\varepsilon \phi \cdot \nabla_\varepsilon u$$

According to [CC1] [CC2], it is easy to check that

$$\begin{cases} \nabla_\varepsilon u = \sqrt{2} \nabla_b u, \\ \Delta_\varepsilon u = 2 \Delta_b u, \end{cases} \tag{13}$$

whenever $u_0 = 0$, i.e.

$$
\begin{aligned}
Lu(x) &:= \Delta_b u(x) - \nabla_b \phi(x) \cdot \nabla_b u(x) \\
&= \frac{1}{2}\left(\Delta_\varepsilon u(x) - \nabla_\varepsilon \phi(x) \cdot \nabla_\varepsilon u(x)\right) \\
&= \frac{1}{2} L_\varepsilon u(x)
\end{aligned} \tag{14}
$$

which implies u is ϕ-pseudoharmonic function on (M^{2n+1}, J, θ) iff u is ϕ-harmonic function on $(M^{2n+1},\ g_\varepsilon)$ whenever $u_0 = 0$ on M.

Now we define the bottom ϕ-spectrums of the weighted Laplacian L and L_ε on (M^{2n+1}, J, θ) and $(M^{2n+1},\ g_\varepsilon)$, respectively. Let $d\mu^\varepsilon = e^{-\phi} dv^\varepsilon$ be the volume form of $(M^{2n+1},\ g_\varepsilon, \phi)$, then variational characterization for bottom ϕ-spectrums λ_1 and λ_1^ε implies

$$
\lambda_1(L) = \inf_{g \in C_0^\infty(M)} \frac{\int_M |\nabla_b g|^2 d\mu}{\int_M g^2 d\mu}
$$

and

$$
\lambda_1^\varepsilon(L_\varepsilon) = \inf_{g \in C_0^\infty(M)} \frac{\int_M |\nabla_b g|^2 d\mu^\varepsilon}{\int_M g^2 d\mu^\varepsilon}.
$$

Moreover, Theorem 3 implies the following application since positive spectrum implies M has infinite volume.

Corollary 5 *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold of infinite ϕ-volume with vanishing torsion and $\phi_0 = 0$. Suppose*

$$
Ric_M(Z, Z) + (\nabla^H)^2 \phi(Z, Z) \geq -k(Z, Z)
$$

for all $Z \in T_{1,0} \oplus T_{0,1}$ and some constant $k \geq 0$. If u is a ϕ-pseudoharmonic function with finite energy on M. Then u is a ϕ-harmonic function with finite energy with respect to the adapted metric g_ε. Moreover, if (M^{2n+1}, J, θ) has at least two ϕ-nonparabolic ends, then $(M^{2n+1},\ g_\varepsilon)$ has at least two ϕ-nonparabolic ends.

Bochner-Type Estimate:

Lemma 6 *Let (M^{2n+1}, J, θ) be a complete pseudohermitian $(2n+1)$ manifold of positive ϕ-spectrum with vanishing torsion and*

$$
Ric_M(Z, Z) + (\nabla^H)^2 \phi(Z, Z) \geq -k(Z, Z)
$$

for all $Z \in T_{1,0} \oplus T_{0,1}$ and some constant $k \geq 0$. If u is a ϕ-pseudoharmonic function with finite energy on M. If we assume $\phi_0 = 0$ and $|\nabla_b \phi| \leq a$ on M for some constant $a \geq 0$. Then for $h = |\nabla_b u| > 0$

$$\tfrac{1}{2} L h^2 \geq \tfrac{2n}{2n-1} |\nabla_b h|^2 - \tfrac{2a}{2n-1} |\nabla_b h| h - k h^2. \tag{15}$$

Proof. It follows from Theorem 3 that u is also a ϕ-harmonic function on $(M^{2n+1}, g_\varepsilon)$ for any $\varepsilon > 0$ with $u_0 = 0$. Applying CR Bochner formula and Riemannian Bochner formula on u, we have

$$\tfrac{1}{2} L |\nabla_b u|^2 = \left| \left(\nabla^H\right)^2 u \right|^2 + 2Ric_M\left((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}\right) + (\nabla^H)^2 \phi\left(\nabla_b u, \nabla_b u\right) \tag{16}$$

and

$$\tfrac{1}{2} L_\varepsilon |\nabla_\varepsilon u|^2_{g_\varepsilon} = \left| (\nabla_\varepsilon)^2 u \right|^2_{g_\varepsilon} + Ric^\varepsilon(L)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right).$$

where

$$\left(\nabla^H\right)^2 \phi\left(\nabla_b u, \nabla_b u\right) = \phi_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} + \phi_{\overline{\alpha\beta}} u_\alpha u_\beta + \phi_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta + \phi_{\overline{\alpha}\beta} u_\alpha u_{\overline{\beta}}$$

and

$$Ric^\varepsilon(L) := Ric^\varepsilon + Hess(\phi).$$

Since $g_\varepsilon = \frac{1}{2} L_\theta + \varepsilon^{-2} \theta \odot \theta$ gives the relation between g_ε and L_θ, we have $\nabla_\varepsilon u = \sqrt{2} \nabla_b u$ and $\Delta_\varepsilon u = 2\Delta_b u$, these imply $|\nabla_\varepsilon u|^2_{g_\varepsilon} = 2|\nabla_b u|^2$ and $\Delta_\varepsilon |\nabla_\varepsilon u|^2_{g_\varepsilon} = 4\Delta_b |\nabla_b u|^2$. Besides, since $\phi_0 = 0$ and $u_0 = 0$, then $\nabla_\varepsilon \phi = \sqrt{2} \nabla_b \phi$ and $L_\varepsilon |\nabla_\varepsilon u|^2_{g_\varepsilon} = 4L|\nabla_b u|^2$. Hence

$$\begin{aligned} &\left| \left(\nabla^H\right)^2 u \right|^2 + 2Ric_M\left((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}\right) + (\nabla^H)^2 \phi\left(\nabla_b u, \nabla_b u\right) \\ &= \frac{1}{4} \left| (\nabla_\varepsilon)^2 u \right|^2_{g_\varepsilon} + \frac{1}{4} Ric^\varepsilon\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) + \frac{1}{4} Hess(\phi)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right). \end{aligned} \tag{17}$$

Since

$$\left(\nabla^H\right)^2 \phi\left(\nabla_b u, \nabla_b u\right) = \frac{1}{4} Hess(\phi)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right). \tag{18}$$

So (18) and $\left[Ric_M + (\nabla^H)^2 \phi\right]\left(\nabla_b u, \nabla_b u\right) \geq -k|\nabla_b u|^2$ give

$$\frac{1}{4} Ric^\varepsilon\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) + \frac{1}{4} Hess(\phi)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) \geq -\left(k + \varepsilon^{-2}\right) |\nabla_b u|^2.$$

In [MW1], they showed Kato's inequality for ϕ-harmonic function u on Riemannian m-manifold with $|\nabla \phi| \leq \alpha$ on M for some constant $\alpha \geq 0$, which is

$$|\nabla^2 u|^2 \geq \tfrac{m}{m-1} |\nabla |\nabla u||^2 - \tfrac{2\alpha}{m-1} |\nabla |\nabla u|| |\nabla u|.$$

Then we have the following Kato's inequality:

$$\left| (\nabla^\varepsilon)^2 u \right|^2_{g_\varepsilon} \geq \tfrac{2n}{2n-1} |\nabla_\varepsilon |\nabla_\varepsilon u||^2 - \tfrac{2\sqrt{2}a}{2n-1} |\nabla_\varepsilon |\nabla_\varepsilon u|| |\nabla_\varepsilon u|,$$

since $|\nabla_\varepsilon \phi| = \sqrt{2}\,|\nabla_b \phi| \le \sqrt{2}a$ on M. Therefore (17) gives

$$\begin{aligned}\left|\left(\nabla^H\right)^2 u\right|^2 &\ge \tfrac{2n}{2n-1}\left|\nabla_b\left|\nabla_b u\right|\right|^2 - \tfrac{2a}{2n-1}\left|\nabla_b\left|\nabla_b u\right|\right|\left|\nabla_b u\right| + \tfrac{1}{4}Ric^\varepsilon\left(L\right)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) \\ &- 2Ric_M\left(\left(\nabla_b u\right)_{\mathbb{C}}, \left(\nabla_b u\right)_{\mathbb{C}}\right) - (\nabla^H)^2\phi\left(\nabla_b u, \nabla_b u\right),\end{aligned}$$

which asserts that (16) can be rewritten as

$$\begin{aligned}\tfrac{1}{2}L\left|\nabla_b u\right|^2 &\ge \tfrac{2n}{2n-1}\left|\nabla_b\left|\nabla_b u\right|\right|^2 - \tfrac{2a}{2n-1}\left|\nabla_b\left|\nabla_b u\right|\right|\left|\nabla_b u\right| + \tfrac{1}{4}Ric^\varepsilon\left(L\right)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) \\ &\ge \tfrac{2n}{2n-1}\left|\nabla_b\left|\nabla_b u\right|\right|^2 - \tfrac{2a}{2n-1}\left|\nabla_b\left|\nabla_b u\right|\right|\left|\nabla_b u\right| - \left(k + \varepsilon^{-2}\right)\left|\nabla_b u\right|^2,\end{aligned}$$

i.e.

$$\tfrac{1}{2}Lh^2 \ge \tfrac{2n}{2n-1}\left|\nabla_b h\right|^2 - \tfrac{2a}{2n-1}\left|\nabla_b h\right| h - \left(k + \varepsilon^{-2}\right) h^2$$

for all $\varepsilon > 0$, where $h = \left|\nabla_b u\right|$. Let $\varepsilon \to \infty$, we obtain (15). ■

Upper bound of $\lambda_1\left(L\right)$:

Lemma 7 *Assume $\left(M^{2n+1}, J, \theta\right)$ is a complete pseudohermitian $\left(2n+1\right)$ manifold with vanishing torsion and and $\phi_0 = 0$ on M. Then*

$$\begin{aligned}&\left(\nabla^H\right)^2\phi\left(\nabla_b u, \nabla_b u\right) = \tfrac{1}{4}Hess(\phi)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) \\ &-\tfrac{\varepsilon}{4}[\phi_\gamma u_{n+\gamma} - \phi_{n+\gamma}u_\gamma - \varepsilon{\omega_i}^\gamma(e_{2n+1})\phi_\gamma u_i - \varepsilon{\omega_i}^{n+\gamma}(e_{2n+1})\phi_{n+\gamma}u_i]u_{2n+1}\end{aligned}$$

for any $u \in C^1\left(M\right)$.

Proof. Since $g_\varepsilon = \frac{1}{2}L_\theta + \varepsilon^{-2}\theta\odot\theta$ gives the relation between g_ε and L_θ, we have $\nabla_\varepsilon u = \sqrt{2}\nabla_b u + \varepsilon u_0$ and $\Delta_\varepsilon u = 2\Delta_b u + \varepsilon^2 u_{00}$. Besides, since $\phi_0 = 0$ then $\nabla_\varepsilon\phi = \sqrt{2}\nabla_b\phi$.

Now set $1 \le i, j \le 2n$, $1 \le a, b, c \le 2n+1$, $1 \le \alpha, \beta, \gamma \le n$, and select $e_j^\varepsilon = e_j$, $e_{2n+1}^\varepsilon = \varepsilon T = \varepsilon e_{2n+1}$, we have

$$\begin{aligned}{\theta_\beta}^\alpha &= {\omega_\beta}^\alpha + i{\omega_\beta}^{n+\alpha} - \left(i\delta_{\alpha\beta}\varepsilon^{-2}\right)\theta \\ {\omega_\gamma}^{2n+1} &= \varepsilon^{-1}\omega^{n+\gamma} \\ {\omega_{n+\gamma}}^{2n+1} &= -\varepsilon^{-1}\omega^\gamma.\end{aligned}$$

Compute, for $e_a^\varepsilon \phi = \phi_a^\varepsilon$,

$$
\begin{aligned}
& Hess(\phi)\left(\nabla_\varepsilon u, \nabla_\varepsilon u\right) \\
& = \left(e_b^\varepsilon e_a^\varepsilon \phi - {\omega_a}^c(e_b^\varepsilon) e_c^\varepsilon \phi\right) u_a^\varepsilon u_b^\varepsilon \\
& = \left(e_b^\varepsilon e_a^\varepsilon \phi - {\omega_a}^\gamma(e_b^\varepsilon)\phi_\gamma^\varepsilon - {\omega_a}^{n+\gamma}(e_b^\varepsilon)\phi_{n+\gamma}^\varepsilon - {\omega_a}^{2n+1}(e_b^\varepsilon) e_{2n+1}^\varepsilon \phi\right) u_a^\varepsilon u_b^\varepsilon \\
& = A + [e_{2n+1}^\varepsilon e_{2n+1}^\varepsilon \phi - {\omega_{2n+1}}^\gamma(e_{2n+1}^\varepsilon)\phi_\gamma - {\omega_{2n+1}}^{n+\gamma}(e_{2n+1}^\varepsilon)\phi_{n+\gamma}]\left(e_{2n+1}^\varepsilon u\right)^2 \\
& + [e_i^\varepsilon e_{2n+1}^\varepsilon \phi - {\omega_{2n+1}}^\gamma(e_i^\varepsilon)\phi_\gamma - {\omega_{2n+1}}^{n+\gamma}(e_i^\varepsilon)\phi_{n+\gamma} \\
& + e_{2n+1}^\varepsilon e_i^\varepsilon \phi - {\omega_i}^\gamma(e_{2n+1}^\varepsilon)\phi_\gamma - {\omega_i}^{n+\gamma}(e_{2n+1}^\varepsilon)\phi_{n+\gamma}]\left(e_{2n+1}^\varepsilon u\right) u_i \\
& = A + \varepsilon^2[\varepsilon^{-1}\omega^{n+\gamma}(\varepsilon e_{2n+1})\phi_\gamma - \varepsilon^{-1}\omega^\gamma(\varepsilon e_{2n+1})\phi_{n+\gamma}] u_{2n+1}^2 \\
& + \varepsilon[\omega^{n+\gamma}(e_i^\varepsilon)\phi_\gamma - \omega^\gamma(e_i^\varepsilon)\phi_{n+\gamma} - {\omega_i}^\gamma(e_{2n+1}^\varepsilon)\phi_\gamma \\
& - {\omega_i}^{n+\gamma}(e_{2n+1}^\varepsilon)\phi_{n+\gamma}] u_{2n+1} u_i \\
& = A + \varepsilon[\phi_\gamma u_{n+\gamma} - \phi_{n+\gamma} u_\gamma - {\omega_i}^\gamma(e_{2n+1}^\varepsilon)\phi_\gamma u_i - {\omega_i}^{n+\gamma}(e_{2n+1}^\varepsilon)\phi_{n+\gamma} u_i] u_{2n+1}
\end{aligned}
$$

where

$$
\begin{aligned}
A = & \left(e_\beta e_\alpha \phi - {\omega_\alpha}^\gamma(e_\beta)\phi_\gamma - {\omega_\alpha}^{n+\gamma}(e_\beta)\phi_{n+\gamma}\right) u_\alpha u_\beta \\
& + \left(e_\beta e_{n+\alpha}\phi - {\omega_{n+\alpha}}^\gamma(e_\beta)\phi_\gamma - {\omega_{n+\alpha}}^{n+\gamma}(e_\beta)\phi_{n+\gamma}\right) u_{n+\alpha} u_\beta \\
& + \left(e_{n+\beta} e_\alpha \phi - {\omega_\alpha}^\gamma(e_{n+\beta})\phi_\gamma - {\omega_\alpha}^{n+\gamma}(e_{n+\beta})\phi_{n+\gamma}\right) u_\alpha u_{n+\beta} \\
& + \left(e_{n+\beta} e_{n+\alpha}\phi - {\omega_{n+\alpha}}^\gamma(e_{n+\beta})\phi_\gamma - {\omega_{n+\alpha}}^{n+\gamma}(e_{n+\beta})\phi_{n+\gamma}\right) u_{n+\alpha} u_{n+\beta}.
\end{aligned}
$$

Besides, for $Z_\alpha = \frac{1}{2}\left(e_\alpha - i e_{n+\alpha}\right)$, $Z_{\overline{\alpha}} = \frac{1}{2}\left(e_\alpha + i e_{n+\alpha}\right)$, then

$$
\begin{aligned}
& \phi_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta + \phi_{\overline{\alpha}\beta} u_\alpha u_{\overline{\beta}} = 2\,\mathrm{Re}\left\{\phi_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta\right\} \\
& = 2\,\mathrm{Re}\left\{\left(Z_{\overline{\beta}} Z_\alpha \phi - \theta_\alpha^\gamma\left(Z_{\overline{\beta}}\right) Z_\gamma \phi\right) Z_{\overline{\alpha}} u Z_\beta u\right\} \\
& = \frac{1}{8}\{\left(e_\beta e_\alpha \phi + e_{n+\beta} e_{n+\alpha}\phi\right)\left(e_\alpha u e_\beta u + e_{n+\alpha} u e_{n+\beta} u\right) \\
& - \left(e_{n+\beta} e_\alpha \phi - e_\beta e_{n+\alpha}\phi\right)\left(e_{n+\alpha} u e_\beta u - e_\alpha u e_{n+\beta} u\right)\} \\
& - \left(\left(\omega_\alpha^\gamma\left(e_\beta\right) - \omega_\alpha^{n+\gamma}\left(e_{n+\beta}\right)\right) e_\gamma \phi + \left(\omega_\alpha^\gamma\left(e_{n+\beta}\right) + \omega_\alpha^{n+\gamma}\left(e_\beta\right)\right) e_{n+\gamma}\phi\right) \\
& \cdot\left(e_\alpha u e_\beta u + e_{n+\alpha} u e_{n+\beta} u\right) \\
& + \left(\left(\omega_\alpha^\gamma\left(e_{n+\beta}\right) + \omega_\alpha^{n+\gamma}\left(e_\beta\right)\right) e_\gamma \phi - \left(\omega_\alpha^\gamma\left(e_\beta\right) - \omega_\alpha^{n+\gamma}\left(e_{n+\beta}\right)\right) e_{n+\gamma}\phi\right) \\
& \cdot\left(e_{n+\alpha} u e_\beta u - e_\alpha u e_{n+\beta} u\right)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\phi_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}}+\phi_{\overline{\alpha}\overline{\beta}}u_{\alpha}u_{\beta}=2\operatorname{Re}\left\{\phi_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}}\right\}\\
&=2\operatorname{Re}\left\{\left(Z_{\beta}Z_{\alpha}\phi-\theta_{\alpha}^{\gamma}\left(Z_{\beta}\right)Z_{\gamma}\phi\right)Z_{\overline{\alpha}}uZ_{\overline{\beta}}u\right\}\\
&=\frac{1}{8}\{\left(e_{\beta}e_{\alpha}\phi-e_{n+\beta}e_{n+\alpha}\phi\right)\left(e_{\beta}ue_{\alpha}u-e_{n+\beta}ue_{n+\alpha}u\right)\\
&+\left(e_{\beta}e_{n+\alpha}\phi+e_{n+\beta}e_{\alpha}\phi\right)\left(e_{\beta}ue_{n+\alpha}u+e_{n+\beta}ue_{\alpha}u\right)\\
&-\left(\left(\omega_{\alpha}^{\gamma}\left(e_{\beta}\right)+\omega_{\alpha}^{n+\gamma}\left(e_{n+\beta}\right)\right)e_{\gamma}\phi+\left(\omega_{\alpha}^{n+\gamma}\left(e_{\beta}\right)-\omega_{\alpha}^{\gamma}\left(e_{n+\beta}\right)\right)e_{n+\gamma}\phi\right)\\
&\cdot\left(e_{\beta}ue_{\alpha}u-e_{n+\beta}ue_{n+\alpha}u\right)\\
&+\left(\left(\omega_{\alpha}^{n+\gamma}\left(e_{\beta}\right)-\omega_{\alpha}^{\gamma}\left(e_{n+\beta}\right)\right)e_{\gamma}\phi-\left(\omega_{\alpha}^{\gamma}\left(e_{\beta}\right)+\omega_{\alpha}^{n+\gamma}\left(e_{n+\beta}\right)\right)e_{n+\gamma}\phi\right)\\
&\cdot\left(e_{\beta}ue_{n+\alpha}u+e_{n+\beta}ue_{\alpha}u\right)\}
\end{aligned}
$$

imply

$$
\begin{aligned}
&\left(\nabla^{H}\right)^{2}\phi\left(\nabla_{b}u,\nabla_{b}u\right)\\
&=\phi_{\alpha\overline{\beta}}u_{\overline{\alpha}}u_{\beta}+\phi_{\overline{\alpha}\beta}u_{\alpha}u_{\overline{\beta}}+\phi_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}}+\phi_{\overline{\alpha}\overline{\beta}}u_{\alpha}u_{\beta}\\
&=\frac{1}{4}\{\left[e_{\beta}e_{\alpha}\phi-\omega_{\alpha}^{\gamma}\left(e_{\beta}\right)e_{\gamma}\phi-\omega_{\alpha}^{n+\gamma}\left(e_{\beta}\right)e_{n+\gamma}\phi\right]e_{\alpha}ue_{\beta}u\\
&+\left[e_{n+\beta}e_{n+\alpha}\phi+\omega_{\alpha}^{n+\gamma}\left(e_{n+\beta}\right)e_{\gamma}\phi-\omega_{\alpha}^{\gamma}\left(e_{n+\beta}\right)e_{n+\gamma}\phi\right]e_{n+\beta}ue_{n+\alpha}u\\
&+\left[e_{\beta}e_{n+\alpha}\phi+\omega_{\alpha}^{n+\gamma}\left(e_{\beta}\right)e_{\gamma}\phi-\omega_{\alpha}^{\gamma}\left(e_{\beta}\right)e_{n+\gamma}\phi\right]e_{n+\alpha}ue_{\beta}u\\
&+\left[e_{n+\alpha}e_{\beta}\phi-\omega_{\alpha}^{\gamma}\left(e_{n+\beta}\right)e_{\gamma}\phi-\omega_{\alpha}^{n+\gamma}\left(e_{n+\beta}\right)e_{n+\gamma}\phi\right]e_{\alpha}ue_{n+\beta}u\}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&\left(\nabla^{H}\right)^{2}\phi\left(\nabla_{b}u,\nabla_{b}u\right)=\tfrac{1}{4}Hess(\phi)\left(\nabla_{\varepsilon}u,\nabla_{\varepsilon}u\right)\\
&-\tfrac{\varepsilon}{4}[\phi_{\gamma}u_{n+\gamma}-\phi_{n+\gamma}u_{\gamma}-\varepsilon\omega_{i}{}^{\gamma}(e_{2n+1})\phi_{\gamma}u_{i}-\varepsilon\omega_{i}{}^{n+\gamma}(e_{2n+1})\phi_{n+\gamma}u_{i}]u_{2n+1}.
\end{aligned}
\tag{19}
$$

■

From this lemma and

$$Ric_{M}\left(Z,Z\right)+(\nabla^{H})^{2}\phi\left(Z,Z\right)\geq-k\left(Z,Z\right)$$

for all $Z\in T_{1,0}\oplus T_{0,1}$ and some constant $k\geq0$, we have

$$\frac{1}{4}Ric_{M}\left(\nabla_{\varepsilon}u,\nabla_{\varepsilon}u\right)+\frac{1}{4}Hess(\phi)\left(\nabla_{\varepsilon}u,\nabla_{\varepsilon}u\right)\geq-\left(k+\varepsilon^{-2}\right)\left|\nabla_{\varepsilon}u\right|^{2}+W$$

where

$$W=-\frac{\varepsilon}{4}[\phi_{\gamma}u_{n+\gamma}-\phi_{n+\gamma}u_{\gamma}-\varepsilon\omega_{i}{}^{\gamma}(e_{2n+1})\phi_{\gamma}u_{i}-\varepsilon\omega_{i}{}^{n+\gamma}(e_{2n+1})\phi_{n+\gamma}u_{i}]u_{2n+1}\geq-c\varepsilon\left|\nabla_{\varepsilon}u\right|^{2}$$

where c is a constant independent of ε.

According the Lemma 2.1 in [MW2],

Lemma 8 *Let* $(M^{2n+1}, g_\varepsilon, \phi)$ *be a complete smooth metric measure space with* $Ric^\varepsilon(L) \geq -k$. *Assume for some nonnegative constants* α *and* β,

$$|\phi(x)| \leq \alpha r(x) + \beta$$

for $x \in M$. *Then there exists a constant* $C > 0$ *such that the volume upper bound*

$$V_\phi(B(R)) \leq Ce^{\left(\sqrt{2nk}+\alpha\right)R}$$

holds for all $R > 1$, *where* $C = C\left(k^{-1/2}, n, \alpha, \beta\right)$.

Proof. Since

$$\left(\frac{J'}{J}\right)'(r) + \frac{1}{2n}\left(\frac{J'}{J}\right)^2(r) + Ric^\varepsilon\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq 0$$

and

$$Ric^\varepsilon\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + \phi''(r) \geq -k$$

then

$$\frac{J'}{J} + \frac{1}{2n}\int_1^r \left(\frac{J'}{J}\right)^2(t)\,dt - \phi'(r) \leq kr + C_0$$

for some constant $C_0 > 0$. Denote

$$h(t) = \frac{J'_\phi}{J_\phi}(t) = \frac{J'}{J}(t) - \phi'(t).$$

Then for any $r \geq 1$,

$$h(r) + \frac{1}{2n}\int_1^r (h(t) + \phi'(t))^2\,dt \leq kr + C_0.$$

Since

$$\int_1^r (h(t) + \phi'(t))^2\,dt \geq (r-1)^{-1}\left(\int_1^r h(t) + \phi'(t)\,dt\right)^2$$

then we obtain

$$h(r) + \frac{1}{2nr}\left(\phi(r) - \phi(1) + \int_1^r h(t)\,dt\right)^2 \leq kr + C_0. \tag{20}$$

Now we claim

$$\int_1^r h(t)\,dt \leq (K+\alpha)r + \alpha + 2\beta + \epsilon C_0$$

for any $r \geq 1$, where $K = \sqrt{2nk}$, $\epsilon = \frac{2n}{K}$.

Define

$$w(r) = (K+\alpha)r + \alpha + 2\beta + \epsilon C_0 - \int_1^r h(t)\,dt.$$

First, one has $w(1) > 0$. Suppose that w is not positive for all $r \geq 1$, and let $R > 1$ be the first number s.t. $w(R) = 0$. Then

$$\int_1^R h(t)\,dt = (K+\alpha)R + \alpha + 2\beta + \epsilon C_0.$$

Besides,

$$\begin{aligned}
&\frac{1}{2nR}\left(\phi(R) - \phi(1) + \int_1^R h(t)\,dt\right)^2 \\
&= \frac{1}{2nR}\left(\phi(R) - \phi(1) + (K+\alpha)R + \alpha + 2\beta + \epsilon C_0\right)^2 \\
&\geq \frac{1}{2nR}\left(KR + \epsilon C_0\right)^2 \\
&\geq \frac{K^2}{2n}R + \frac{K}{n}\epsilon C_0
\end{aligned}$$

Taking this into (20), one has

$$h(R) + \frac{K^2}{2n}R + \frac{K}{n}\epsilon C_0 \leq kR + C_0$$

i.e.

$$h(r) \leq \left(k - \frac{K^2}{2n}\right)R + \left(1 - \frac{K}{n}\epsilon\right)C_0.$$

If we select $\epsilon = \frac{2n}{K}$. This shows that

$$w'(r) = (K+\alpha) - h(R) > 0$$

i.e. $\exists\, \delta > 0$ s.t. $w(R-\delta) \leq w(R) = 0$. Then this gives a contradiction to the choice of R.

Now by (20), we have

$$\log J_\phi(r) - \log J_\phi(1) \leq \left(\sqrt{2nk} + \alpha\right)r + \alpha + 2\beta + \sqrt{\frac{2n}{k}}C_0.$$

Hence for $R \geq 1$,

$$V_\phi(B(R)) \leq Ce^{\left(\sqrt{2nk}+\alpha\right)R}.$$

■

Theorem 9 *Let $(M^{2n+1}, g_\varepsilon, \phi)$ be a complete smooth metric measure space with $Ric^\varepsilon(L) \geq -k$. Assume the linear growth rate of ϕ is α. Then we have*

$$\lambda_1^\varepsilon(L) \leq \frac{1}{4}\left(\sqrt{2nk} + \alpha\right)^2.$$

In particular, if ϕ is of sublinear growth, then

$$\lambda_1^{\varepsilon}(L) \leq \frac{nk}{2}.$$

Proof. Let ψ be a cut-off function on $B(R)$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } r(x) \leq R-1, \\ 0 & \text{if } r(x) \geq R, \end{cases}$$

and

$$0 \leq \psi(x) \leq 1 \text{ and } |\nabla \psi| \leq 2 \text{ on } M.$$

Set $\Psi(x) = \psi(x) \exp\left(-\frac{\sqrt{2nk}+\alpha}{2} + \delta\right)$ as a test function in the variational principle for $\lambda_1^{\varepsilon}(L)$, where $\delta > 0$ is a constant. Then by Lemma 8, we obtain

$$\lambda_1^{\varepsilon}(L) \leq \frac{1}{4}\left(\sqrt{2nk} + \alpha + \delta\right)^2.$$

Since δ is arbitrary, this completes the proof. ■

Now we show that positive spectrum of $(M^{2n+1}, J, \theta, \phi)$ implies $(M^{2n+1}, g_{\varepsilon}, \phi)$ has positive spectrum.

Lemma 10 *Suppose $(M^{2n+1}, J, \theta, \phi)$ is a complete noncompact pseudohermitian $(2n+1)$-manifold with positive spectrum $\lambda_1 > 0$, then $(M^{2n+1}, g_{\varepsilon}, \phi)$ has positive spectrum $\lambda_1^{\varepsilon} \geq 2\lambda_1 > 0$.*

Proof. If $f \in W_0^{1,2}(M^{2n+1})$, by using $dv^{\varepsilon} = \frac{1}{2^n n! \varepsilon} \theta \wedge (d\theta)^n$, then we have

$$\frac{\int_M |\nabla_{\varepsilon} f|^2 d\mu^{\varepsilon}}{\int_M |f|^2 d\mu^{\varepsilon}} = \frac{\int_M \left(2|\nabla_b f|^2 + \varepsilon^2 f_0^2\right) d\mu}{\int_M |f|^2 d\mu} = 2\frac{\int_M |\nabla_b f|^2 d\mu}{\int_M |f|^2 d\mu} + \varepsilon^2 \frac{\int_M f_0^2 d\mu}{\int_M |f|^2 d\mu}.$$

Hence

$$2\lambda_1 = 2 \inf_{f \in W_0^{1,2}(M^{2n+1})} \frac{\int_M |\nabla_b f|^2 d\mu}{\int_M |f|^2 d\mu} \leq \frac{\int_M |\nabla_{\varepsilon} f|^2 d\mu^{\varepsilon}}{\int_M |f|^2 d\mu^{\varepsilon}}$$

and then we conclude that $\lambda_1^{\varepsilon} \geq 2\lambda_1 > 0$ where λ_1^{ε} is the spectrum of $(M^{2n+1}, g_{\varepsilon}, \phi)$. ■

Since Lemma 10 implies $2\lambda_1(L) \leq \lambda_1^{\varepsilon}(L_{\varepsilon})$, then by choosing $\alpha = \sqrt{2}a$, and letting ε tends to infinity, we have the following corollary.

Corollary 11 *If (M^{2n+1}, J, θ) is a complete pseudohermitian $(2n+1)$ manifold with positive ϕ-spectrum, vanishing torsion and $[2Ric(L) - Tor(L)](Z, Z) \geq -2k|Z|^2$ for all $Z \in T_{1,0}$ and $\phi_0 = 0$ on M. Assume the linear growth rate of ϕ is a, then*

$$\lambda_1(L) \leq \frac{1}{8}\left(\sqrt{4nk + \varepsilon + \varepsilon^{-1}} + \sqrt{2}a\right)^2.$$

In particular, if ϕ is of sublinear growth, then

$$\lambda_1(L) \le \frac{n(2n-1)}{4}.$$

Paneitz operator :

In this work, we always assume u is a complex-value C^∞ function on M.

Definition 12 *The second-order differential operator*

$$B_{\alpha\overline{\beta}}u = u_{\alpha\overline{\beta}} - \frac{u_{\gamma\overline{\gamma}}}{n}h_{\alpha\overline{\beta}}$$

is a trace-free operator.

Definition 13 *The third-order differential operator*

$$Pu = P_\alpha u\theta^\alpha = \left(u_{\overline{\beta}\beta\alpha} + niA_{\alpha\beta}u_{\overline{\beta}}\right)\theta^\alpha$$

is called the Paneitz operaotr.

Definition 14 *Let*

$$P_0 u = 4\left(\delta_b(Pu) + \overline{\delta}_b\left(\overline{P}u\right)\right)$$

where $\delta_b\left(u_\beta\theta^\beta\right) = u_{\beta\overline{\beta}}$ is the divergence operator that takes $(1,0)$ forms to the functions.

Proposition 15 *For complex-value function u,*

$$\begin{aligned} P_0 u &= 2\left(\Delta_b^2 u + n^2T^2u - 2n\,\mathrm{Re}\,Qu\right) \\ &= 2\Box_b\overline{\Box}_b u - 4nQu \quad \left(= 8\overline{\delta}_b\left(\overline{P}u\right)\right) \\ &= 2\overline{\Box}_b\Box_b u - 4n\overline{Q}u \quad \left(= 8\delta_b(Pu)\right), \end{aligned}$$

where

$$\Box_b u = (-\Delta_b + inT)u = -2u_{\overline{\alpha}\alpha}.$$

Moreover,

$$\begin{aligned} \left[\Box_b, \overline{\Box}_b\right]u &= 2n\left(Q - \overline{Q}\right)u \\ &= 4niJQ \\ &= 4ni\left(\left(A_{\overline{\alpha\beta}}u_\alpha\right)_{,\beta} + \overline{\left(A_{\overline{\alpha\beta}}u_\alpha\right)_{,\beta}}\right). \end{aligned}$$

is pure image.

Proof. Since $Qu = 2i\left(A_{\overline{\alpha\beta}}u_{\alpha}\right)_{,\beta}$ implies

$$\begin{aligned}
P_0 u &= 4\left(\delta_b\left(Pu\right) + \overline{\delta}_b\left(\overline{P}u\right)\right) \\
&= 4\left(u_{\overline{\beta}\beta\alpha\overline{\alpha}} + ni\left(A_{\alpha\beta}u_{\overline{\beta}}\right)_{,\overline{\alpha}} + u_{\beta\overline{\beta}\overline{\alpha}\alpha} - ni\left(A_{\overline{\alpha\beta}}u_{\beta}\right)_{,\alpha}\right) \\
&= 4\left(u_{\overline{\beta}\beta\alpha\overline{\alpha}} + u_{\beta\overline{\beta}\overline{\alpha}\alpha}\right) - 2n\left(\overline{Q}u + Qu\right);
\end{aligned}$$

therefore, due to

$$\begin{aligned}
u_{\overline{\beta}\beta\alpha\overline{\alpha}} + u_{\beta\overline{\beta}\overline{\alpha}\alpha} &= u_{\beta\overline{\beta}\alpha\overline{\alpha}} - inu_{0\alpha\overline{\alpha}} + u_{\beta\overline{\beta}\overline{\alpha}\alpha} \\
&= \Delta_b\left(u_{\beta\overline{\beta}}\right) - inu_{0\alpha\overline{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
u_{\overline{\beta}\beta\alpha\overline{\alpha}} + u_{\beta\overline{\beta}\overline{\alpha}\alpha} &= u_{\overline{\beta}\beta\alpha\overline{\alpha}} + u_{\overline{\beta}\beta\overline{\alpha}\alpha} + inu_{0\overline{\alpha}\alpha} \\
&= \Delta_b\left(u_{\overline{\beta}\beta}\right) + inu_{0\overline{\alpha}\alpha},
\end{aligned}$$

we have

$$\begin{aligned}
P_0 u &= 2\left(\Delta_b\left(\Delta_b u\right) - inu_{0\alpha\overline{\alpha}} + inu_{0\overline{\alpha}\alpha}\right) - 2n\left(\overline{Q}u + Qu\right) \\
&= 2\left(\Delta_b\left(\Delta_b u\right) - in\left(u_{0\alpha\overline{\alpha}} - u_{0\overline{\alpha}\alpha}\right)\right) - 2n\left(\overline{Q}u + Qu\right) \\
&= 2\left(\Delta_b^2 u + n^2 u_{00}\right) - 2n\left(\overline{Q}u + Qu\right).
\end{aligned}$$

Moreover, it is clear that $\Box_b u = -2u_{\overline{\alpha}\alpha}$ implies

$$\begin{aligned}
P_0 u &= 4\left(u_{\overline{\beta}\beta\alpha\overline{\alpha}} + u_{\beta\overline{\beta}\overline{\alpha}\alpha}\right) - 2n\left(\overline{Q}u + Qu\right) \\
&= \overline{\Box}_b\Box_b u + \Box_b\overline{\Box}_b u - 2n\left(\overline{Q}u + Qu\right).
\end{aligned}$$

Then, by (21),

$$\begin{aligned}
\overline{\Box}_b\Box_b u &= \left(\Delta_b + inT\right)\left(\Delta_b - inT\right)u \\
&= \left(\Delta_b^2 - in\Delta_b T + inT\Delta_b + n^2T^2\right)u \\
&= \left(\Delta_b^2 - in\left[\Delta_b, T\right] + n^2T^2\right)u \\
&= \left(\Delta_b^2 + in\left[\Delta_b, T\right] + n^2T^2\right)u - 2in\left[\Delta_b, T\right]u \\
&= \left(\Delta_b - inT\right)\left(\Delta_b + inT\right)u - 4inJQu \\
&= \Box_b\overline{\Box}_b u - 2n\left(Q - \overline{Q}\right)u
\end{aligned}$$

gives

$$4\delta_b\left(Pu\right) = \overline{\Box}_b\Box_b u - 2n\overline{Q} = \Box_b\overline{\Box}_b u - 2nQ = 4\overline{\delta}_b\left(\overline{P}u\right)$$

and

$$P_0 u = 2\Box_b\overline{\Box}_b u - 4nQu, \quad P_0 u = 2\overline{\Box}_b\Box_b u - 4n\overline{Q}.$$

■

Let purely holomorphic 2nd-order operator $Qu = 2i\left(A_{\overline{\alpha\beta}}u_{\alpha}\right)_{,\beta}$, then

Lemma 16

$$[\Delta_b, T]u = 2JQu \tag{21}$$

where

$$JQu = \operatorname{Im} Qu = \frac{Qu - \overline{Q}u}{2i} = \left(A_{\overline{\alpha\beta}}u_\alpha\right)_{,\beta} + \overline{\left(A_{\overline{\alpha\beta}}u_\alpha\right)_{,\beta}}$$

i.e.

$$Ji\,(X) = \operatorname{Re} X.$$

Since $\partial_b u = u_\alpha \theta^\alpha$ and $\overline{\partial}_b u = u_{\overline{\alpha}}\theta^{\overline{\alpha}}$, then the formal adjoint of ∂_b on functions (with respect to the Levi form) is $\partial_b^* = -\delta_b$, therefore,

$$\textstyle\int_M \langle Pu, \partial_b v\rangle = 8\int_M P_\alpha u \overline{v}_{\overline{\alpha}} = -\int_M \left(P_\alpha u\right)_{,\overline{\alpha}} \overline{v} = -\int_M \left(P_0 u\right)\overline{v}.$$

Proposition 17 *P_0 is a self-adjoint operator in $L^2(M)$.*

Proof. For any complex-value functions u and v,

$$\begin{aligned}\langle Qu, v\rangle_{L^2} &= \int_M (Qu)\,\overline{v}dv = 2i\int_M \left(A_{\overline{\alpha\beta}}u_\alpha\right)_{,\beta}\overline{v}dv \\ &= 2i\int_M A_{\overline{\alpha\beta}}u_\alpha\overline{v}_\beta dv \\ &= 2i\int_M u\left(A_{\overline{\alpha\beta}}\overline{v}_\beta\right)_{,\alpha} dv \\ &= \int_M uQ\overline{v}dv \\ &= \left\langle u, \overline{Q}v\right\rangle_{L^2},\end{aligned}$$

then $Q + \overline{Q}$ is symmetric on $L^2(M)$. Then $P_0 u = 2\left(\Delta_b^2 u + n^2T^2u - n\left(Q + \overline{Q}\right)\right)$ implies

$$\langle P_0 u, v\rangle_{L^2} = \langle u, P_0 v\rangle_{L^2}.$$

■

Lemma 18

$$\int_M \left\langle \overline{Q}u, u\right\rangle = -\int_M Tor\left(\left(\nabla_b u\right)_{\mathbb{C}}, \left(\nabla_b u\right)_{\mathbb{C}}\right) \quad ????$$

Proof. Since

$$\langle Qu,u\rangle_{L^2}=\int_M (Qu)\,\overline{u}dv=2i\int_M \left(A_{\overline{\alpha\beta}}u_\alpha\right)_{,\beta}\overline{u}dv$$
$$=2i\int_M A_{\overline{\alpha\beta}}u_\alpha\overline{u}_\beta dv$$

and

$$\langle Qu,v\rangle_{L^2}=\left\langle u,\overline{Q}v\right\rangle_{L^2},$$

where

$$Tor\,(X,Y)=i\left(A_{\overline{\alpha\beta}}X^{\overline{\alpha}}Y^{\overline{\beta}}-A_{\alpha\beta}X^\alpha Y^\beta\right)=2\operatorname{Re}\left(iA_{\overline{\alpha\beta}}X^{\overline{\alpha}}Y^{\overline{\beta}}\right)$$

■

Proposition 19

$$\tfrac{n-1}{n}P_\alpha u=\left(B_{\alpha\overline{\beta}}u\right)_{,\beta} \tag{22}$$

and

$$\textstyle\sum_{\alpha,\beta}\int_M\left|B_{\alpha\overline{\beta}}u\right|^2dv=\frac{n-1}{n}\int_M (P_0u)\,\overline{u}dv \tag{23}$$

Proof. Since

$$\begin{array}{rcl}\left(B_{\alpha\overline{\beta}}u\right)_{,\beta} &=& \left(u_{\alpha\overline{\beta}}-\frac{u_{\gamma\overline{\gamma}}}{n}h_{\alpha\overline{\beta}}\right)_{,\beta}\\ &=& \left(u_{\overline{\beta}\alpha}+ih_{\alpha\overline{\beta}}u_0-\frac{u_{\overline{\gamma}\gamma}+inu_0}{n}h_{\alpha\overline{\beta}}\right)_{,\beta}\\ &=& u_{\overline{\beta}\alpha\beta}+iu_{0\alpha}-\frac{u_{\overline{\gamma}\gamma\alpha}+inu_{0\alpha}}{n}\\ &=& u_{\overline{\beta}\beta\alpha}+inA_{\alpha\rho}u_{\overline{\rho}}-ih_{\alpha\overline{\beta}}A_{\beta\rho}u_{\overline{\rho}}-\frac{u_{\overline{\gamma}\gamma\alpha}}{n}\\ &=& u_{\overline{\beta}\beta\alpha}+i\,(n-1)\,A_{\alpha\rho}u_{\overline{\rho}}-\frac{u_{\overline{\gamma}\gamma\alpha}}{n}\\ &=& \frac{n-1}{n}\left(u_{\overline{\beta}\beta\alpha}+inA_{\alpha\beta}u_{\overline{\beta}}\right).\end{array}$$

then (22) follows by $P_\alpha u=u_{\overline{\beta}\beta\alpha}+niA_{\alpha\beta}u_{\overline{\beta}}$.

The trace-free property of $B_{\alpha\overline{\beta}}u$ implies

$$\begin{aligned}\left|B_{\alpha\overline{\beta}}u\right|^2&=\left(B^{\alpha\overline{\beta}}u\right)\left(B_{\alpha\overline{\beta}}\overline{u}\right)\\ &=\left(u^{\alpha\overline{\beta}}-\frac{u_{\overline{\gamma}\gamma}}{n}h^{\alpha\overline{\beta}}\right)\left(B_{\alpha\overline{\beta}}\overline{u}\right)\\ &=u^{\alpha\overline{\beta}}\left(B_{\alpha\overline{\beta}}\overline{u}\right)\\ &=\left(u^\alpha B_{\alpha\overline{\beta}}\overline{u}\right)^{,\overline{\beta}}-u^\alpha\left(B_{\alpha\overline{\beta}}\overline{u}\right)^{,\overline{\beta}}\\ &=\left(u^\alpha B_{\alpha\overline{\beta}}\overline{u}\right)^{,\overline{\beta}}-\frac{n-1}{n}u^\alpha P_\alpha\overline{u}\\ &=\left(u^\alpha B_{\alpha\overline{\beta}}\overline{u}\right)^{,\overline{\beta}}-\frac{n-1}{n}\left(uP_\alpha\overline{u}\right)^{,\alpha}+\frac{n-1}{n}u\left(P_\alpha\overline{u}\right)^{,\alpha}\\ &=\left(u^\alpha B_{\alpha\overline{\beta}}\overline{u}\right)^{,\overline{\beta}}-\frac{n-1}{n}\left(uP_\alpha\overline{u}\right)^{,\alpha}+\frac{n-1}{n}u\left(P_0\overline{u}\right),\end{aligned}$$

therefore, by self-adjoint property of P_0, we obtain

$$\sum_{\alpha,\beta}\int_M \left|B_{\alpha\overline{\beta}}u\right|^2 dv = \frac{n-1}{n}\int_M (P_0u)\,\overline{u}dv.$$

■

Proposition 20

$$\begin{aligned}\int_M (P_0u)\,\overline{u}dv &= \int_M 2\left\langle\overline{\Box}_b u, \Box_b u\right\rangle - 4n\left\langle Qu, u\right\rangle dv = \int_M 2u_{\alpha\overline{\alpha}}\overline{u}_{\alpha\overline{\alpha}} + 8niA_{\overline{\alpha\beta}}u_\alpha\overline{u}_\beta dv \\ &= \int_M 2\left\langle\Box_b u, \overline{\Box}_b u\right\rangle - 4n\left\langle\overline{Q}u, u\right\rangle dv = \int_M 2u_{\overline{\alpha}\alpha}\overline{u}_{\overline{\alpha}\alpha} - 8niA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}}dv\end{aligned}$$

Proof. Since

$$\begin{aligned}\int_M \left\langle\Box_b u, v\right\rangle &= \int_M \overline{v}\Box_b u = -2\int_M \overline{v}\partial_b\overline{\partial}_b u = 2\int_M \partial_b\overline{v}\overline{\partial}_b u \\ &= -2\int_M \overline{\partial}_b\partial_b\overline{v}u \\ &= \int_M \overline{\Box}_b\overline{v}u = \int_M \left\langle u, \Box_b v\right\rangle\end{aligned}$$

then

$$\begin{aligned}\int_M (P_0u)\,\overline{u}dv &= \int_M \left\langle 2\Box_b\overline{\Box}_b u - 4nQu, u\right\rangle dv \\ &= \int_M 2u_{\alpha\overline{\alpha}}\overline{u}_{\alpha\overline{\alpha}} - 4n\left\langle Qu, u\right\rangle dv \\ &= \int_M 2u_{\alpha\overline{\alpha}}\overline{u}_{\alpha\overline{\alpha}} + 8niA_{\overline{\alpha\beta}}u_\alpha\overline{u}_\beta dv\end{aligned}$$

and

$$\begin{aligned}\int_M (P_0u)\,\overline{u}dv &= \int_M \left\langle 2\overline{\Box}_b\Box_b u - 4n\overline{Q}u, u\right\rangle dv \\ &= \int_M 2\left\langle\Box_b u, \overline{\Box}_b u\right\rangle - 4n\left\langle\overline{Q}u, u\right\rangle dv \\ &= \int_M 2\left\langle\Box_b u, \overline{\Box}_b u\right\rangle + 8ni\left\langle\left(A_{\alpha\beta}u_{\overline{\alpha}}\right)_{,\overline{\beta}}, u\right\rangle dv \\ &= \int_M 2u_{\overline{\alpha}\alpha}\overline{u}_{\overline{\alpha}\alpha} - 8niA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}}dv\end{aligned}$$

■

Eigenvalue of Peneitz operator :

Proposition 21

$$\int u_{\overline{\alpha}\beta}\overline{u}_{\alpha\overline{\beta}} = \int_M \frac{n-1}{n}\left(P_0 u\right)\overline{u} + \frac{1}{4n}\left\langle \Box_b u, \Box_b u\right\rangle \tag{24}$$

$$\int u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} = \int \frac{1}{4}\left\langle \Box_b u, \Box_b u\right\rangle + iu_\alpha \overline{u}_{0\overline{\alpha}} - n\left|u_0\right|^2 - iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} + R^\rho{}_\alpha u_{\overline{\alpha}}\overline{u}_\rho \tag{25}$$

Proof. Since

$$\sum_{\alpha,\beta}\int_M \left|B_{\alpha\overline{\beta}}u\right|^2 dv = \frac{n-1}{n}\int_M \left(P_0 u\right)\overline{u}dv$$

implies

$$\begin{aligned}
\int u_{\overline{\alpha}\beta}\overline{u}_{\alpha\overline{\beta}} &= \int \left(u_{\overline{\alpha}\beta} - \frac{u_{\overline{\rho}\rho}}{n}h_{\overline{\alpha}\beta}\right)\left(\overline{u}_{\alpha\overline{\beta}} - \frac{\overline{u}_{\gamma\overline{\gamma}}}{n}h_{\alpha\overline{\beta}}\right) + \frac{\overline{u}_{\gamma\overline{\gamma}}}{n}u_{\overline{\alpha}\alpha} \\
&= \int B_{\overline{\alpha}\beta}u\overline{B_{\overline{\alpha}\beta}u} + \frac{1}{4n}\left\langle \Box_b u, \Box_b u\right\rangle \\
&= \int_M \frac{n-1}{n}\left(P_0 u\right)\overline{u} + \frac{1}{4n}\left\langle \Box_b u, \Box_b u\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\int u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} &= \int u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\beta\alpha} = -\int u_{\overline{\alpha}}\overline{u}_{\beta\alpha\overline{\beta}} = -\int u_{\overline{\alpha}}\overline{u}_{\beta\overline{\beta}\alpha} + iu_{\overline{\alpha}}\overline{u}_{\alpha 0} + R_\beta{}^\rho{}_{\alpha\overline{\beta}}u_{\overline{\alpha}}\overline{u}_\rho \\
&= \int u_{\overline{\alpha}\alpha}\overline{u}_{\beta\overline{\beta}} + iu_{\overline{\alpha}}\left(\overline{u}_{0\alpha} - A_{\alpha\beta}\overline{u}_{\overline{\beta}}\right) + R^\rho{}_\alpha u_{\overline{\alpha}}\overline{u}_\rho \\
&= \int \frac{1}{4}\left\langle \Box_b u, \Box_b u\right\rangle + iu_{\overline{\alpha}}\overline{u}_{0\alpha} - iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} + R^\rho{}_\alpha u_{\overline{\alpha}}\overline{u}_\rho \\
&= \int \frac{1}{4}\left\langle \Box_b u, \Box_b u\right\rangle - iu_{\overline{\alpha}\alpha}\overline{u}_0 - iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} + R^\rho{}_\alpha u_{\overline{\alpha}}\overline{u}_\rho \\
&= \int \frac{1}{4}\left\langle \Box_b u, \Box_b u\right\rangle - i\left(u_{\alpha\overline{\alpha}} - inu_0\right)\overline{u}_0 - iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} + R^\rho{}_\alpha u_{\overline{\alpha}}\overline{u}_\rho \\
&= \int \frac{1}{4}\left\langle \Box_b u, \Box_b u\right\rangle + iu_\alpha\overline{u}_{0\overline{\alpha}} - n\left|u_0\right|^2 - iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} + R^\rho{}_\alpha u_{\overline{\alpha}}\overline{u}_\rho
\end{aligned}$$

■

Proposition 22 *For any complex-valued function u, we have*

$$\begin{aligned}
-\frac{1}{2}\Box_b|\overline{\partial}_b u|^2 = &\sum_{\alpha,\beta}(u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} + u_{\overline{\alpha}\beta}\overline{u}_{\alpha\overline{\beta}}) + Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \\
&- \frac{1}{2n}\langle\overline{\partial}_b u, \overline{\partial}_b\Box_b u\rangle - \frac{n+1}{2n}\langle\overline{\partial}_b\Box_b u, \overline{\partial}_b u\rangle \\
&- \frac{1}{n}\langle\overline{P}u, \overline{\partial}_b u\rangle + \frac{n-1}{n}\langle P\overline{u}, \partial_b\overline{u}\rangle,
\end{aligned} \tag{26}$$

where $(\nabla_b u)_{\mathbb{C}} = \sum_\alpha u_{\overline{\alpha}}Z_\alpha$ is the corresponding complex $(1,0)$-vector filed of $\nabla_b u$.

Lemma 23 *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold, for any complex smooth function u,*

$$\begin{aligned} \tfrac{n+1}{4n}\int \langle \Box_b u, \Box_b u\rangle &= \int \sum_{\alpha,\beta} u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} + \tfrac{1}{n}\int (P_0 u)\,\overline{u} \\ &\quad + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}). \end{aligned} \tag{27}$$

Proof. By integrating the Bochner formula (26), we obtain

$$\begin{aligned} 0 &= \int \sum_{\alpha,\beta}(u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} + u_{\overline{\alpha}\beta}\overline{u}_{\alpha\overline{\beta}}) + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \\ &\quad - \frac{1}{2n}\int \langle \overline{\partial}_b u, \overline{\partial}_b \Box_b u\rangle - \frac{n+1}{2n}\int \langle \overline{\partial}_b \Box_b u, \overline{\partial}_b u\rangle \\ &\quad - \frac{1}{n}\int \langle \overline{P}u, \overline{\partial}_b u\rangle + \frac{n-1}{n}\int \langle P\overline{u}, \partial_b \overline{u}\rangle \\ &= \int \sum_{\alpha,\beta}(u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} + u_{\overline{\alpha}\beta}\overline{u}_{\alpha\overline{\beta}}) - \frac{n+2}{4n}\int \langle \Box_b u, \Box_b u\rangle \\ &\quad + \frac{2-n}{n}\int (P_0 u)\,\overline{u} + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}). \end{aligned}$$

On the other hand, by (24), we have

$$\int u_{\overline{\alpha}\beta}\overline{u}_{\alpha\overline{\beta}} = \int_M \frac{n-1}{n}(P_0 u)\,\overline{u} + \frac{1}{4n}\langle \Box_b u, \Box_b u\rangle .$$

Substituting this into the above integral formula, we get

$$\begin{aligned} \frac{n+1}{4n}\int \langle \Box_b u, \Box_b u\rangle &= \int \sum_{\alpha,\beta} u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} + \frac{1}{n}\int (P_0 u)\,\overline{u} \\ &\quad + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}). \end{aligned}$$

■

Proposition 24 *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with*

$$Ric(Z, Z) \geq k\,|Z|^2 . \tag{28}$$

In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of the Kohn Laplacian $\Box_b$ must satisfy

$$\lambda_1 \geq \frac{2nk}{n+1}. \tag{29}$$

Remark 25 *Moreover, if the equality of 29 holds, the corresponding eigenfunction u will satisfy*

$$u_{\overline{\alpha}\overline{\beta}} = 0, \tag{30}$$

$$Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) = k|\overline{\partial}_b u|^2. \tag{31}$$

Proof. Let u be an eigenfunction of the Kohn Laplacian with respect to the first nonzero eigenvalue λ_1, that is, $\Box_b u = \lambda_1 u$. By integration formula of the Bochner formula (27), we obtain,

$$\begin{array}{rcl} \frac{n+1}{2n}\lambda_1 \int |\overline{\partial}_b u|^2 & = & \int \sum\limits_{\alpha,\beta} u_{\overline{\alpha}\overline{\beta}} \overline{u}_{\alpha\beta} + \frac{1}{n}\int (P_0 u)\,\overline{u} \\ & & + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \\ & \geq & k \int |\overline{\partial}_b u|^2, \end{array}$$

where we have used, for $n = 1$,

$$\int (P_0 u)\,\overline{u} \geq 0.$$

It follows that $\lambda_1 \geq \frac{2nk}{n+1}$. Moreover, if $\lambda_1 = \frac{2nk}{n+1}$, then $u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} = 0$. ■

Theorem 26 *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with*

$$Ric(Z, Z) \geq -k\,|Z|^2\,.$$

If λ_1 is the first nonzero eigenvalue of the Kohn Laplacian, and Λ_1 is the first nonzero eigenvalue of the Paneitz operator, then we have

$$\Lambda_1 \leq n\lambda_1 \left(\frac{n+1}{2n}\lambda_1 + k\right).$$

Proof. Let u be an eigenfunction of the Kohn Laplacian with respect to the first nonzero eigenvalue λ_1, that is, $\Box_b u = \lambda_1 u$. By integration formula of the Bochner formula (27), we obtain

$$\begin{array}{rcl} \frac{n+1}{2n}\lambda_1 \int |\overline{\partial}_b u|^2 & = & \int \sum\limits_{\alpha,\beta} u_{\overline{\alpha}\overline{\beta}} \overline{u}_{\alpha\beta} + \frac{1}{n}\int (P_0 u)\,\overline{u} \\ & & + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \\ & \geq & \frac{\Lambda_1}{n\lambda_1} \int |\overline{\partial}_b u|^2 + k \int |\overline{\partial}_b u|^2, \end{array}$$

then we have

$$\frac{n+1}{2n}\lambda_1 \geq +k$$

and

$$\Lambda_1 \leq n\lambda_1 \left(\frac{n+1}{2n}\lambda_1 - k\right).$$

■

Theorem 27 *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with*

$$\left(Ric - \frac{1}{4}Tor\right)(Z, Z) \leq -k\,|Z|^2\,.$$

If λ_1 is the first nonzero eigenvalue of the Kohn Laplacian, and Λ_1 is the first nonzero eigenvalue of the Paneitz operator, then we have

$$\Lambda_1 \geq 2n\lambda_1 k.$$

Proof. By (25) and 27,

$$\begin{array}{rcl} \frac{n+1}{4n}\int \langle \Box_b u, \Box_b u\rangle &=& \int \sum\limits_{\alpha,\beta} u_{\overline{\alpha}\overline{\beta}}\overline{u}_{\alpha\beta} + \frac{1}{n}\int (P_0 u)\,\overline{u} \\ && +\int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \\ &=& \int \frac{1}{4}\langle \Box_b u, \Box_b u\rangle + iu_{\overline{\alpha}}\overline{u}_{0\alpha} - n\,|u_0|^2 - iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} + R^{\rho}{}_{\alpha}u_{\overline{\alpha}}\overline{u}_{\rho} \\ && +\frac{1}{n}\int (P_0 u)\,\overline{u} + \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \end{array}$$

then one has

$$\frac{1}{n}\int (P_0 u)\,\overline{u} = \int \frac{1}{4n}\langle \Box_b u, \Box_b u\rangle + iu_{\overline{\alpha}\alpha}\overline{u}_0 + n\,|u_0|^2 + iA_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} - 2R^{\rho}{}_{\alpha}u_{\overline{\alpha}}\overline{u}_{\rho}. \tag{32}$$

Since it implies

$$\frac{1}{n}\int (P_0 \overline{u})\,u = \int \frac{1}{4n}\langle \overline{\Box}_b u, \overline{\Box}_b u\rangle + i\overline{u}_{\overline{\alpha}\alpha}u_0 + n\,|u_0|^2 + iA_{\alpha\beta}\overline{u}_{\overline{\alpha}}u_{\overline{\beta}} - 2R^{\rho}{}_{\alpha}\overline{u}_{\overline{\alpha}}u_{\rho}$$

and this equals to (we select conjugate on the both sides)

$$\frac{1}{n}\int (P_0 u)\,\overline{u} = \int \frac{1}{4n}\langle \overline{\Box}_b u, \overline{\Box}_b u\rangle - iu_{\alpha\overline{\alpha}}\overline{u}_0 + n\,|u_0|^2 - iA_{\overline{\alpha}\overline{\beta}}u_{\alpha}\overline{u}_{\beta} - 2R^{\overline{\rho}}{}_{\overline{\alpha}}u_{\alpha}\overline{u}_{\overline{\rho}} \tag{33}$$

then take the sum of (32) and (33),

$$\begin{aligned} \frac{2}{n}\int (P_0\overline{u})\,u = \frac{1}{4n}\int \langle \Box_b u, \Box_b u\rangle &+ \langle \overline{\Box}_b u, \overline{\Box}_b u\rangle \\ &+ (2n+1)\int |u_0|^2 + i\int \left(A_{\alpha\beta}u_{\overline{\alpha}}\overline{u}_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_{\alpha}\overline{u}_{\beta}\right) \\ &- 4\int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) \end{aligned} \tag{34}$$

here we use

$$\int i\left(u_{\overline{\alpha}\alpha} - u_{\alpha\overline{\alpha}}\right)\overline{u}_0 = \int u_0\overline{u}_0 = \int |u_0|^2$$

Hence

$$\int (P_0 u)\,\overline{u} \geq -2n \int Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) - \frac{1}{4} i \left(A_{\alpha\beta} u_{\overline{\alpha}} \overline{u}_{\overline{\beta}} - A_{\overline{\alpha\beta}} u_\alpha \overline{u}_\beta \right)$$
$$\geq 2nk \int \left| \overline{\partial}_b u \right|^2 \geq 2nk\lambda_1 \int |u|^2$$

whenever

$$Ric((\nabla_b u)_{\mathbb{C}}, (\nabla_b u)_{\mathbb{C}}) - \frac{1}{4} i \left(A_{\alpha\beta} u_{\overline{\alpha}} \overline{u}_{\overline{\beta}} - A_{\overline{\alpha\beta}} u_\alpha \overline{u}_\beta \right) \leq -k \left| \overline{\partial}_b u \right|^2$$

If we select Λ is the first eigenvalueof P_0 and u is the eigenfunction, then we have

$$\Lambda \geq 2nk\lambda_1.$$

■

Theorem 28 *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with*

$$Ric(Z, Z) = -k\,|Z|^2 \text{ and } Tor(Z, Z) = 0.$$

If λ_1 is the first nonzero eigenvalue of the Kohn Laplacian, and Λ_1 is the first nonzero eigenvalue of the Paneitz operator, then we have

$$\lambda_1 \geq \frac{6nk}{n+1}.$$

Proof. By Theorem 26 and Theorem 27, we obtain

$$2nk\lambda_1 \leq \Lambda_1 \leq \frac{n+1}{2} (\lambda_1)^2 - nk\lambda_1$$

and theorem follows. ■

Remark 29 *As the Theorem 28, we have $\Lambda_1 \geq \frac{12n^2k^2}{n+1}$.*

(四). 參考文獻:

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附件五

科技部補助專題研究計畫執行國際合作與移地研究心得報告

日期：107 年 12 月 25 日

計畫編號	MOST　106-2115-M-003-009 -		
計畫名稱	維騰拉普拉斯算子和擬埃爾米特瑞奇曲率		
出國人員姓名	陳瑞堂	服務機構及職稱	國立臺灣師範大學數學系副教授
出國時間	107 年 7 月 16 日 至 107 年 7 月 23 日	出國地點	浙江金華浙江師範大學
出國研究目的	□實驗 □田野調查 □採集樣本 ☑移地研究 □使用國外研究設施		

一、執行國際合作與移地研究過程

7 月 16 日和吳進通教授由桃園機場至杭州蕭山機場，轉車到杭州進行訪問和移地研究，過兩天在和邱教授會合，7 月 23 日由蕭山機場回國。

二、研究成果

1. CR Yau's gradient estimate for Witten Laplacian via Barkry-Emery pseudohermitian Ricci curvature.

2. Almost Schur lemma on smooth metric measure spaces.

106年度專題研究計畫成果彙整表

計畫主持人：陳瑞堂					**計畫編號：**106-2115-M-003-009-		
計畫名稱：維騰拉普拉斯算子和擬埃爾米特瑞奇曲率							
成果項目					量化	單位	質化（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文			0	篇	
		研討會論文			0		
		專書			0	本	
		專書論文			0	章	
		技術報告			0	篇	
		其他			1	篇	未發表論文
	智慧財產權及成果	專利權	發明專利	申請中	0	件	
				已獲得	0		
			新型/設計專利		0		
		商標權			0		
		營業秘密			0		
		積體電路電路布局權			0		
		著作權			0		
		品種權			0		
		其他			0		
	技術移轉	件數			0	件	
		收入			0	千元	
國外	學術性論文	期刊論文			0	篇	
		研討會論文			0		
		專書			0	本	
		專書論文			0	章	
		技術報告			0	篇	
		其他			0	篇	
	智慧財產權及成果	專利權	發明專利	申請中	0	件	
				已獲得	0		
			新型/設計專利		0		
		商標權			0		
		營業秘密			0		
		積體電路電路布局權			0		
		著作權			0		
		品種權			0		
		其他			0		

	技術移轉	件數	0	件	
		收入	0	千元	
參與計畫人力	本國籍	大專生	0	人次	
		碩士生	3		兼任研究助理
		博士生	0		
		博士後研究員	0		
		專任助理	0		
	非本國籍	大專生	0		
		碩士生	0		
		博士生	2		兼任研究助理
		博士後研究員	0		
		專任助理	0		
其他成果（無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）					

科技部補助專題研究計畫成果自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現（簡要敘述成果是否具有政策應用參考價值及具影響公共利益之重大發現）或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以100字為限）

　　□實驗失敗

　　□因故實驗中斷

　　□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形（請於其他欄註明專利及技轉之證號、合約、申請及洽談等詳細資訊）

論文：□已發表　■未發表之文稿　□撰寫中　□無

專利：□已獲得　□申請中　■無

技轉：□已技轉　□洽談中　■無

其他：（以200字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性，以500字為限）

這個工作我們建構特徵值的估計，還有一些延續性結果可以再加以討論。

4. 主要發現

本研究具有政策應用參考價值：■否　□是，建議提供機關

（勾選「是」者，請列舉建議可提供施政參考之業務主管機關）

本研究具影響公共利益之重大發現：□否　□是

說明：（以150字為限）