科技部補助專題研究計畫報告

Harary圖上的 k元全支配問題(第3年)

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本研究具有政策應用參考價值：■否 □是，建議提供機關
（勾選「是」者，請列舉建議可提供施政參考之業務主管機關）
本研究具影響公共利益之重大發現：□否 □是

中 華 民 國 109 年 11 月 15 日

中文摘要：我們研究 Harary 圖 $H_{m, n}$ 上的二元全支配數。我們完成了 $m=3$ 之分析，並給了部分 $m=5$ 的結果。此外，研究並展示當 m 足夠大時，存在 m 正則圖其二元全支配數與下界 $\lceil 2n/m\rceil$ 吻合。

中文關鍵詞：二元全支配問題
Harary 圖

英文摘要：In this report, we are concerned with the 2-tuple total domination number $\gamma_{\times2,t}(H_{m, n})$ of Harary graphs $H_{m, n}$, where m is odd. We complete the analysis for $m=3$, and give some result for $m=5$. For m large enough, we show that there exists m-regular graphs whose 2-tuple total domination number matches the lower bound of $\lceil 2n/m\rceil$.

英文關鍵詞：2-tuple total domination
Harary graph

Abstract

In this report, we are concerned with the 2-tuple total domination number $\gamma_{\times 2,t}(H_{m,n})$ of Harary graphs $H_{m,n}$, where m is odd. We complete the analysis for $m = 3$, and give some result for $m = 5$. For m large enough, we show that there exists m-regular graphs whose 2-tuple total domination number matches the lower bound of $\lceil 2n/m \rceil$.

Keywords: k-tuple total domination; Harary graph

Contents

1 Introduction

Given a graph G, a vertex subset is a *k-tuple total dominating set* if every vertex has at least k neighbors in the set, and the size of a minimum k-tuple total dominating set (kTDS for abbreviation) is called the *k-tuple total domination number* of G. The notion of k-tuple total domination was proposed by Henning and Kazemi [3] in 2010, and some results of this problem were proposed recently. This project is proposed due to the work of Kazemi and Pahlavsay [6]. They are concerned with the problem with $k = 2$, where the problem is also called the *double total domination problem*. They gave upper bounds on the double total domination number of Harary graphs, by a standard constructive way. However, for some cases, there is still a gap of 1 between the given upper bound and a known lower bound. This motivates us to propose this project. In the next section, we give a brief survey for this problem.

1.1 Previous results of the k-tuple total domination problem

We start with the k-tuple total domination problem, and then get into the specific work of Kazemi and Pahlavsay [6]. The k-tuple total domination was proposed by Henning and Kazemi [3] in 2010. In their paper, they gave some results for general graphs. In particular, they consider the class of complete p-partite graphs and the k join of graphs.[1] Later on, they gave some results on the product of graphs. We summarize the known results in Table 1.

Henning and Kazemi [4] gave a lower bound on $\gamma_{\times k,t}$, which is widely applied in the subsequent research. We summarize it in Proposition 1.

Proposition 1 (Henning and Kazemi [4]). *Let G be a graph of order n, and let Δ and δ be the maximum and minimum degree of G, respectively. If $\delta \geq k$, we have*

$$\gamma_{\times k,t}(G) \geq \left\lceil \frac{kn}{\Delta} \right\rceil .$$

Usually, the k-tuple total domination problem serves as a generalization of the total domination problem, i.e. the k-tuple total domination problem with $k = 1$. To make the work nontrivial, k is assumed to be at least two, and the case in which $k = 2$, the double total domination problem, is usually independently discussed. Related results are summarized in Table 2.

Recently, Pradhan [9], and Lan and Chang [8] investigated the problem from an algorithmic viewpoint. The complexity of computing the k-tuple total domination number on different graph classes are conducted. Results are shown in Table 3.

[1] The k join of graphs G and H is a graph obtained from the disjoint union of G and H with each vertex of G joining at least k vertices of H.

[2] The complementary prism of a graph G is the graph formed from the disjoint union of G and its complement $\bar{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\bar{G}$.

Table 1: Previous results for the k-tuple total domination problem.

Authors	Year	Graph class	Results
Henning, Kazemi [3]	2010	general	$\gamma_{\times k,t}(G) = \tau_k(H_G)$
			$\gamma_{\times k,t}(G) \leq \left(p + \sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-i}\right) n$, $0 \leq p \leq 1$
		fixed k, sufficiently large δ	$\gamma_{\times k,t}(G) \leq (\ln\delta + (k-1+o(1))\ln\ln\delta)\, n/\delta$
		k join of a graph and K_{k+1}	$k+1$
		complete p-partite, $p \geq k+1$	$\gamma_{\times k,t}(G) = k+1$
		complete p-partite, $p = k$	$\gamma_{\times k,t}(G) = k+2$
		complete p-partite, $n_i \geq \lceil kp/(p-1)\rceil$, $p < k$	$\gamma_{\times k,t}(G) = \lceil kp/(p-1)\rceil$
Kazemi [5]	2011	complementary prisms of complete p-partite graphs[2]	the number differs according to the size of the partite sets.
Henning, Kazemi [4]	2012	product of complete m- and n-partite graphs	$\gamma_{\times k,t}(G) = k+2$, if $n \geq k+2$; $k+3$ if $n = k+1$
		$K_m \times K_n$	$\gamma_{\times k,t}(G) = k+2$, if $n \geq k+2$; $k+3$ if $n = k+1$

1.2 Double total domination on Harary graphs

A Harary graph $H_{m,n}$ is defined depending on two positive integers m and n with $m < n$. The vertex set of $H_{m,n}$ is the integers, ranging from 1 to n, and the edge set is defined depending on the parities of m and n. Usually, it is convenient to define the graph from a geometric viewpoint. The n vertices are placed around a circle, equally spaced. If m is even, each vertex is adjacent to its closest m vertices. Otherwise, if n is even, each vertex is adjacent to its closest $m-1$ vertices and to its diametrically opposite vertex. If both n and m are odd, each vertex is adjacent to its

[3]Given $n \not\equiv 0 \mod 4$ and $m \not\equiv 0 \mod 4$,

$$\alpha(m,n) = \begin{cases} \frac{m(n+1)}{2}, & \text{if } n \text{ is odd,} \\ \frac{n(m+1)}{2}, & \text{if } n \equiv 2 \mod 4,\ m \text{ is odd, and } n \leq 2m, \\ \frac{m(n+2)}{2}, & \text{otherwise.} \end{cases}$$

Table 2: Previous results for double total domination problem.

Authors	Year	Graph class	Results
Henning, Kazemi [3]	2010	general	$\gamma_{\times 2,t}(G) \leq (\ln(\delta+2)+\ln\delta+1)\, n/\delta$
		$\Delta(G) \leq n/2$	$\gamma_{\times 2,t}(G) = n$
		bipartite graphs with a partite set of vertices of minimum degree	$\gamma_{\times 2,t}(G) \leq 9n/10$
		cubic bipartite	$\gamma_{\times 2,t}(G) \leq 8n/9$
Kazemi [5]	2011	complementary prism of cycles	$\gamma_{\times 2,t}(G) = n+2, n \geq 5$
Henning, Kazemi [4]	2012	$C_m \times C_n$	$\gamma_{\times 2,t}(G) = mn/2$, if $m \equiv n \equiv 0 \mod 4$; $\gamma_{\times 2,t}(G) \leq \alpha(m,n)$, otherwise[3]
Kazemi, Pahlavsay [6]	2016	Harary graphs	See Sec. 1.2

closest $m-1$ vertices, and for a specific set of consecutive $n+1$ vertices, say 1 to $n+1$, each of them has a clockwise n-step neighbor. An example is given in Figure 1.

A Harary graph $H_{m,n}$ is an m-connected graph of order n with the minimum number of edges [2]. On Harary graphs, Kazemi and Pahlavsay [6] gave the following results for the double total domination problem.

Theorem 1 (Kazemi and Pahlavsay [6]). *If both m and n are even, then*

$$\gamma_{\times 2,t}(H_{m,n}) = \left\lceil \frac{2n}{m} \right\rceil.$$

If m is odd and n is even, then

$$\left\lceil \frac{2n}{m} \right\rceil \leq \gamma_{\times 2,t}(H_{m,n}) \leq \left\lceil \frac{2n}{m} \right\rceil + 1.$$

If both m and n are odd, then

$$\left\lceil \frac{2n-1}{m} \right\rceil \leq \gamma_{\times 2,t}(H_{m,n}) \leq \left\lceil \frac{2n-1}{m} \right\rceil + 1.$$

The theorem was derived as follows. The number $\gamma_{\times 2,t}(H_{m,n})$ is lower bounded using Proposition 1. For the upper bound, Kazemi and Pahlavsay applied two strategies to build a double total dominating set:

- Each vertex is dominated by its clockwise/counterclockwise neighbor within $m/2$ steps.
- Some of the chosen vertices are dominated by their diametrically opposite neighbors.

Table 3: Algorithmic results for the k-tuple total domination problem.

Authors	Year	Graph class	Results
Pradhan [9]	2012	split graphs	NP-c
		doubly chordal graphs	NP-c
		bipartite graphs	NP-c, APX-c for $\Delta(G) = k+2$
		chordal bipartite graphs	P
		split graphs	NP-complete
Lan, Chang [8]	2014	each block is a clique, a cycle, or a complete bipartite graph	linear time
		undirected path graph	NP-c

For the two kinds of double total dominating sets, the one with fewer vertices gives the upper bound. We note here that if m and n satisfy certain conditions, the upper bound matches the lower bound.[4] However, for succinctness, we omit the details.

2 Results on Harary graphs

We give the exact values for $\gamma_{\times 2,t}(H_{3,2n})$ and $\gamma_{\times 2,t}(H_{3,2n+1})$ in Sections 2.3 and 2.2, respectively,

In the following, we write n as $2n$ and $2n+1$ to emphasize that n is even and odd, respectively.

2.1 Even n and $m = 3$

Kazemi and Pahlavsay [6] gave the following result.[5]

Proposition 2 (Kazemi and Pahlavsay [6]). *Let $2n = 3\ell + r$ with $r < 3$. Then*

$$\left\lceil \frac{4n}{3} \right\rceil \leq \gamma_{\times 2,t}(H_{3,2n}) \leq \left\lceil \frac{4n}{3} \right\rceil + 1.$$

In particular, $\gamma_{\times 2,t}(H_{3,2n}) = \lceil \frac{4n}{3} \rceil$ if $r \neq 1$.

To complete the analysis of $\gamma_{\times 2,t}(H_{3,2n})$, we assume $r = 1$. Let S be a 2TDS of $H_{3,2n}$. By Proposition 2,

$$|S| \geq \left\lceil \frac{4n}{3} \right\rceil = \left\lceil 2\ell + \frac{2}{3} \right\rceil = 2\ell + 1.$$

The following lemma gives a necessary condition for $|S| = 2\ell + 1$.

Lemma 2. *Assume that $2n = 3\ell+1$. If $|S| = 2\ell+1$, then there is exactly one vertex v satisfying $|N(v) \cap S| = 3$ and $v \in V \setminus S$.*

[4]Please refer to Theorems 2.3 and 2.4 in [6].

[5]The result given by Kazemi and Pahlavsay is more general, namely, for $H_{2m+1,2n}$ with $m < n$. For succinctness we summarize the case $2m + 1 = 3$ only.

Proof. Since $2 \cdot 2n < 3 \cdot (2\ell + 1)$, there exists exactly one such vertex v. Consider the subgraph induced by S. By the handshaking lemma, $v \notin S$. □

Based on Lemma 2, we develop the following theorem.

Theorem 3. *Assume* $2n = 3\ell + 1$. *Then*

$$\gamma_{\times 2,t}(H_{3,2n+1}) = \begin{cases} 2\ell + 1, & \text{if } \ell = 1 \mod 4, \\ 2\ell + 2, & \text{if } \ell = 3 \mod 4. \end{cases}$$

2.1.1 Proofs

Lemma 4. *Assume* $2n = 3\ell + 1$ *and* $n \geq 8$. *Let* S *and* S' *be 2-tuple total dominating sets of* $H_{3,2n}$ *and* $H_{3,2n-12}$, *respectively. Then*

$$|S| = 2\ell + 1 \implies |S'| = 2\ell + 1.$$

Proof. Without loss of generality, let $2n \notin S$ and $|N(2n) \cap S| = 3$. Then $n \in S$. Moreover,

$$S \cap \{i, n+i\colon\ 1 \leq i \leq 7\} = \{1, 2, 3, 4, 7, n+1, n+4, n+5, n+6, n+7\}.$$

S induces a 2TDS of $H_{3,2n-12}$ by removing the 12 vertices, $\{i, n+i\colon\ 1 \leq i \leq 6\}$. □

Lemma 5. *For* $2n = 3\ell + 1$ *and* $\ell \equiv 3 \mod 4$, $\gamma_{\times 2,t}(H_{3,2n}) = 2\ell + 2$.

Proof. By Proposition 5, it suffices to show $\gamma_{\times 2,t}(H_{3,2n}) \neq 2\ell + 1$. Let $\ell = 4k + 3$ for $k \geq 0$. We prove by induction on k. The inductive step is shown in Lemma 4. We verify the base case in the following. Namely, we claim $\gamma_{\times 2,t}(H_{3,10}) \neq 7$.

Suppose that there is a 2TDS S of size 7. Assume without loss of generality that $10 \in S$. Then

$$\{1, 5, 9\} \subseteq S \implies \{4, 6\} \subseteq S \implies \{3, 7\} \cap S = \emptyset \implies |N(2) \cap S| < 2,$$

a contradiction. □

Lemma 6. *For* $2n = 3\ell + 1$ *and* $\ell \equiv 1 \mod 4$, $\gamma_{\times 2,t}(H_{3,2n}) = 2\ell + 1$.

Proof. Let S be the requested 2TDS, and let $2n$ be the vertex v with $|N(v) \cap S| = 3$. Then

$$S = \left\{6k + i\colon\ 1 \leq i \leq 4, 0 \leq k < \frac{n-2}{6}\right\} \cup \left\{n + 6k + i\colon\ 4 \leq i \leq 7, 0 \leq k < \frac{n-2}{6}\right\} \cup \{n-1, n, n+1\}.$$

It can be easily verified that S is a 2TDS and $|S| = 2\ell + 1$. □

2.2 Odd n and $m = 3$

From Proposition 1, the 2-tuple total domination number of $H_{2m+1,2n+1}$ is bounded below by $\frac{2n+1}{m+1}$. The bound can be improved according to the following inequality. Let S be a 2-tuple total dominating set of $H_{2m+1,2n+1}$. Then

$$|S| \cdot (2m+1) + 1 \geq 2 \cdot (2n+1), \tag{1}$$

and thus

$$\gamma_{\times 2,t}(H) \geq \left\lceil \frac{4n+1}{2m+1} \right\rceil.$$

Similar to $\gamma_{\times 2,t}(H_{3,2n})$, Kazemi and Pahlavsay [6] gave the following result.[6]

Proposition 3 (Kazemi and Pahlavsay [6])**.** *Assume $2n+1 = 3\ell + r$ with $r < 3$.*

$$\left\lceil \frac{4n+1}{3} \right\rceil \leq \gamma_{\times 2,t}(H_{3,2n+1}) \leq \left\lceil \frac{4n+1}{3} \right\rceil + 1.$$

In particular, $\gamma_{\times 2,t}(H_{3,2n+1}) = \lceil \frac{4n+1}{3} \rceil$ if $r = 1$.

Proposition 4 (Yang and Wang [12])**.** *Assume $2n+1 = 3\ell$. Then $\gamma_{\times 2,t}(H_{3,2n+1}) = \lceil \frac{4n+1}{3} \rceil$.*

Note that $\lceil \frac{4n+1}{3} \rceil = 2\ell + 1$. To complete the analysis of $\gamma_{\times 2,t}(H_{3,2n+1})$, it remains to consider $2n+1 = 3\ell + 2$. The result we develop is stated in Theorem 7.

Theorem 7. *Assume $2n+1 = 3\ell + 2$. Then*

$$\gamma_{\times 2,t}(H_{3,2n+1}) = 2\ell + 1.$$

First, consider vertex $n+1$, i.e. the vertex with degree 4.

Lemma 8. *Assume $2n+1 = 3\ell + 2$. Let S be a 2-tuple total dominating set of $H_{3,2n+1}$. Then*

$$|S| = 2\ell + 1 \implies n+1 \in S.$$

Proof. Otherwise, $3 \cdot |S| < 4n + 2$, a contradiction. □

Specifically, a 2-tuple total dominating set exists only if

$$\forall_{v \in V} \quad |N(v) \cap S| = 2. \tag{2}$$

The proofs are based on the necessity established above.

2.2.1 Proofs of Theorem 7

Proof. Let

$$S = \{1, n+1, 2n+1\} \cup \{3k+i, n+3k+i \colon\ 1 \leq k \leq \frac{n-2}{3}, i \in \{0,1\}\}.$$

S is a 2-tuple total dominating set of size $2\ell + 1$. See also Fig. 1.

□

[6] As mentioned previously, their result is more general.

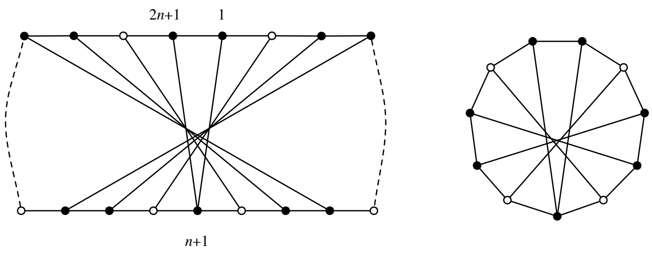


Figure 1: The labelling principle and a minimum 2TDS of $H_{3,11}$.

2.3 Even n and $m = 5$

We write $2n = 5\ell + r$ with $0 \le r < 5$. Kazemi and Pahlavsay [6] proved the following.

Proposition 5 (Kazemi and Pahlavsay [6]). *Let $2n = 5\ell + r$ with $r < 5$. Then*

$$\left\lceil \frac{4n}{5} \right\rceil \le \gamma_{\times 2,t}(H_{5,2n}) \le \left\lceil \frac{4n}{5} \right\rceil + 1.$$

In particular, $\gamma_{\times 2,t}(H_{5,2n}) = \lceil \frac{4n}{5} \rceil$ if $r = 0$ or $2 < r < 5$.

Note that $\lceil \frac{4n}{5} \rceil = 2\ell + 1$.

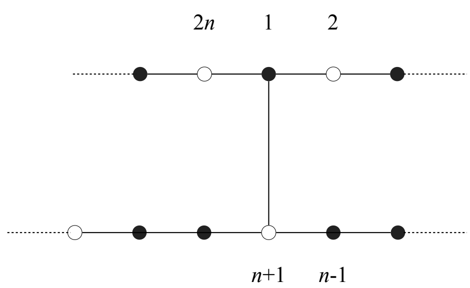
To derive the exact value of $\gamma_{\times 2,t}(H_{5,2n})$, it remains to consider the cases $r = 1$ and $r = 2$. From Proposition 5, we derive the following.

Proposition 6. *Let D be a 2TDS of $H_{5,2n}$ with $2n = 5\ell + r$ and $1 \le r \le 2$. If $|D| = 2\ell + 1$, then*

$$|\{u \in V \colon\ |N(u) \cap D| = 2\}| \ge 2n - 3.$$

Moreover, there is a vertex $x \in V \setminus D$ such that $|N(x) \cap D| = 3$. In particular, when $r = 2$ the vertex x is unique.

Proof. The first part of the proposition can be proved using double counting, as how Proposition 5 is proved. In addition, by considering the subgraph induced by D, it can be derived that at least one vertex of degree 3 or 5 is in $V \setminus D$. To prove the proposition, it suffices to verify that $\deg(x) \ne 5$. Suppose to the contrary that $\deg(x) = 5$. Without loss of generality, let $x = n + 1$. There is only one set of vertices, the black ones in the following figure, that can be the 2TDS.



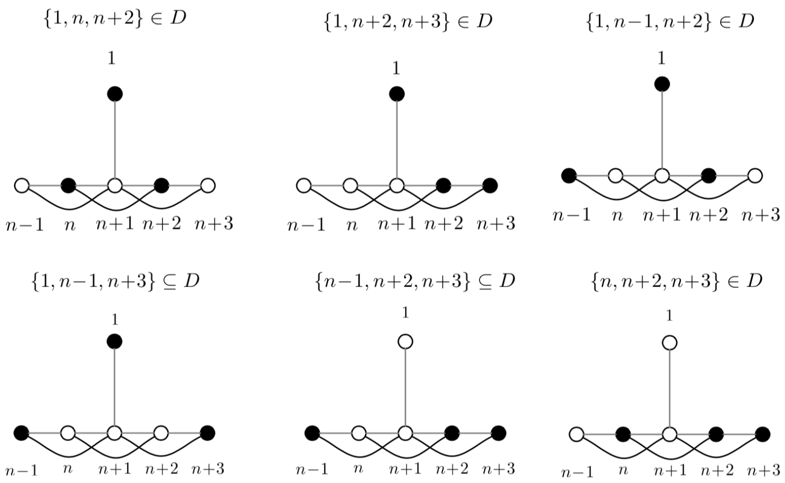
Note that when $\deg(x) = 5$, all the other vertices are of degree 2. However, the existence of a requested 2TDS leads to the existence of vertices of degree 3 (vertices 2 and $2n$ in the figure). □

In the following, for a 2TDS D we call the vertices x with $|N(x) \cap D| \geq 2$ the *singular* vertices.

2.3.1 $r = 2$

Lemma 9. *For $2n = 5\ell + 2$, $\gamma_{\times 2,t}(H_{5,2n}) = 2\ell + 2$.*

Proof. Suppose to the contrary that $\gamma_{\times 2,t}(H_{5,2n}) = 2\ell + 1$. By Proposition 6 there is a unique singular vertex x, with $x \in V \setminus D$ and $|N(x) \cap D| = 3$. Without loss of generality assume that $x = n + 1$. Because of symmetry, it suffices to consider the six cases, with black vertices indicating the members of D:

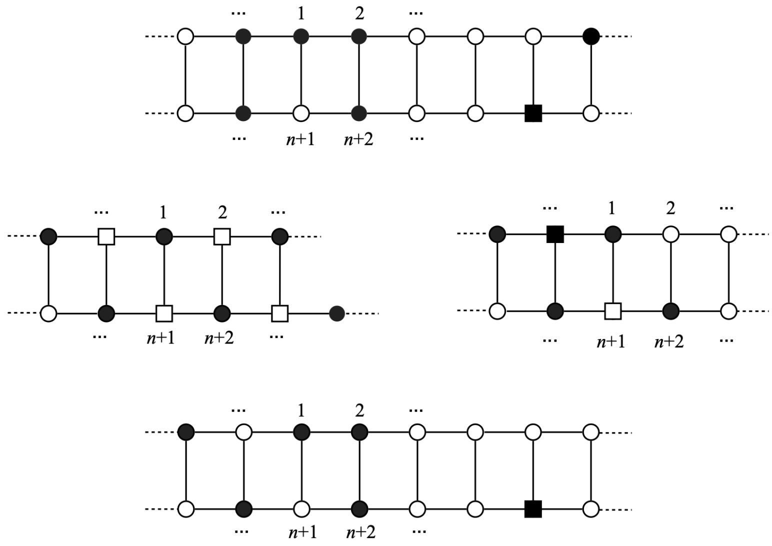


In either case, there will be an additional vertex y with $|N(y) \cap D| = 3$. □

2.3.2 $r = 1$

Recall that $2n = 5\ell + 1$. The smallest case is $H_{5,6} = K_6$, and $\gamma_{\times 2,t}(H_{5,6}) = 2\ell + 1 = 3$. As we show below, this is the only case in which $\gamma_{\times 2,t}(H_{5,2n} = 2\ell + 1$. In the following, we assume that ℓ is an odd number greater than or equal to 3, and show that $\gamma_{\times 2,t}(H_{5,2n}) = 2\ell + 2$.

Let D be a 2TDS, and suppose to the contrary that $|D| = 2\ell + 1$. By Proposition 6, there is a vertex v satisfying $|N(v) \cap D| = 3$. Without loss of generality, we assume that $v = n + 1$, as the six cases depicted in the proof of Lemma 9. It can be verified that $|N(1) \cap D| \neq 4$ by an immediate case-by-case analysis. Thus it remains to consider $|N(1) \cap D| = 2$ and $|N(1) \cap D| = 3$. Below is an illustration for $N(1) \cap D = \{2, 2n\}$, in which squared vertices indicate those violating Proposition 6.



Remark 1. *The analysis for* $|N(1) \cap D| = 2$ *and* $|N(1) \cap D| = 3$ *is still done by enumerating all possibilities and analyze case by case. The whole process is tedious, so we do not include it here. Details can be found in [10, 11].*

The only case that needs a nontrivial analysis is given in Lemma 10.

Lemma 10. *Let* D *be a 2TDS of* $H_{5,2n}$*, where* $2n = 5\ell + 1$ *and* ℓ *is an odd number at least* 3*. If* $|D| = 2\ell + 1$ *and* $n + 1 \notin D$ *with* $|N(n+1)| = 3$*, then, in a symmetric manner,* $1 \notin D$*,* $N(1) = \{2, 3, 2n-1\}$*, and* $N(n+1) = \{n-2, n+2, n+3\}$*.*

By Lemma 10, we may prove that $H_{5,2n} = 2\ell + 2$ for $\ell \geq 3$, which is done by induction on the number of vertices. We start with the induction basis.

Lemma 11. $\gamma_{\times 2,t}(H_{5,16}) = 8$*.*

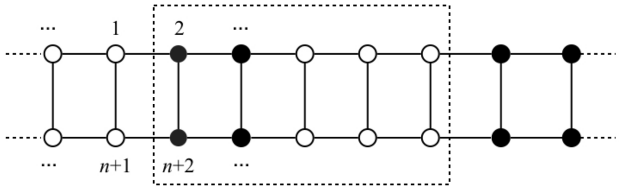
Proof. By Proposition 6 it suffices to show that $\gamma_{\times 2,t}(H_{5,16}) \neq 7$. Suppose to the contrary that $\gamma_{\times 2,t}(H_{5,16}) = 7$. Let D be a minimum 2TDS of $H_{3,16}$ with, without loss of generality, vertex 9 having three neighbors in D. It can be derived that vertex 1 has three neighbors in D also. Let $N(1) \cap D = \{2, 3, 15\}$ and $N(9) \cap D = \{7, 10, 11\}$. Since every vertex has degree 5 and $|N(u) \cap D| = 2$ for $u \in V(H_{3,10}) \setminus \{1, 9\}$, the vertex z is the uniquely singular and $|N(u) \cap D| = 3$, we have

$$\begin{aligned} 10 \in D &\overset{N(2)}{\Longrightarrow} \{4\} \notin D \\ \{2,3\} \in D &\overset{N(4)}{\Longrightarrow} \{5,6,12\} \notin D \\ \{3,10\} \in D &\overset{N(11)}{\Longrightarrow} 13 \notin D \\ 7 \in D &\overset{N(15)}{\Longrightarrow} 14 \in D. \end{aligned}$$

Clearly, this leads to the fact that $12 \notin D$ and $|N(12) \cap D| = |\{4, 10, 11, 14\}| = 4$, a contradiction. Thus, $\gamma_{\times 2,t}(H_{5,16}) = 8$. $\square$

Lemma 12. *For $2n = 5\ell + 1$, $\gamma_{\times 2,t}(H_{5,2n}) = 2\ell + 2$.*

Proof. By Lemma 11, this holds for $\ell = 3$. Let n be the least number such that $\gamma_{\times 2,t}(H_{5,2n+10}) = 2(\ell + 2) + 1$. By Lemma 10 the elements in the corresponding 2TDS of $H_{5,2n+10}$ are the black dots in the following figure.



Note that by removing the vertices in the dashed-square, and adding the edges $\{1, 7\}$, $\{1, 8\}$, $\{n + 1, n + 7\}$, $\{n + 1, n + 8\}$, $\{2n, 7\}$ and $\{n - 1, n + 7\}$, we have the graph $H_{5,2n}$, with black dots forming a 2TDS of size $2\ell + 1$, which is a contradiction. □

3 Double total domination on regular graphs

Given a simple undirected graph G, consider the *neighborhood hypergraph* of G. Namely, the hypergraph (V, F) with $V = V(G)$ and

$$F = \{N(v) \colon v \in V(G)\},$$

where $N(v)$ is the open neighborhood of v in G. A *transversal* T of a hypergraph H is a vertex subset such that for $A \in F$, $T \cap A \neq \varnothing$. For the double total domination problem (i.e. the 2-tuple total domination problem) it asks for the minimum transversal T satisfying $|T \cap A| \geq 2$ for every $A \in F$.

A k-coloring of a hypergraph $H = (V, F)$ is a mapping $f \colon V \to [k]$. For a k-coloring f, an edge A is *monochromatic* if $|f(v) \colon v \in A| = 1$. A coloring of hypergraph H is *proper* if no edge of H is monochromatic. Finding the minimum transversal of a hypergraph H can be interpreted as finding a specific proper k-coloring of H.

Note that in the Harary graph $H_{m,n}$. The neighborhood hypergraph is m-uniform and m-regular. Moreover, each edge intersects exactly $2m - 1$ other edges.

3.1 Probabilistic analysis – the first attempt

Our attempt is to apply Lovász local lemma to see under which conditions there exists an $2n$-vertex $(2m + 1)$-regular graph that has the double total domination number $\lceil 4n/(2m + 1)\rceil$.

Theorem 13 (Asymmetric Lovász local lemma []). *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a finite set of events. For $A \in \mathcal{A}$ let $\Gamma(A)$ denote set of events such that $A \notin \Gamma(A)$ and events in $\mathcal{A} \setminus \Gamma(A)$ are mutually independent. If there exists an assignment of reals $x : \mathcal{A} \to [0, 1)$ such that*

$$\forall A \in \mathcal{A} : \Pr(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B))$$

then $\Pr\left(\overline{A_1} \wedge \cdots \wedge \overline{A_n}\right) \geq \prod_{i \in \{1,\cdots,n\}} (1 - x(A_i))$.

Theorem 14 (Symmetric Lovász local lemma [])**.** *If $ep(d+1) \leq 1$, where e is the base of natural logarithms, then the probability that none of the events occurs is nonzero.*

Let $f\colon V \to [k]$ be a coloring, and let color 1 denote the membership to the double total dominating set. Namely, $f(v) = 1$ if and only if v is a member of the corresponding double total dominating set. We color the vertices using k colors uniformly at random. Let A_i be the event in which

$$|v \in F_i\colon\ f(v) = 1| \neq 2.$$

Let $x = 2m + 1$. Then

$$\Pr(A_i) \quad = \quad 1 - \binom{x}{2} \frac{1}{k^2} \left(\frac{k-1}{k}\right)^{x-2}.$$

Let $p = \Pr(A_i)$. If $ep(d+1) < 1$, where A_i is mutually independent to all but at most d events, then Lovász local lemma shows that $\Pr(\cap \overline{A_i}) > 0$. Here $d = 2x - 1$. However, for $0 < q < 1$

$$e\left(1 - \binom{x}{2} q^2 (1-q)^{x-2}\right)(2x) \geq 1.$$

The Lovász local lemma does not apply to this situation.

The probability to the individual event A_i is too large. (It approaches to 1 as x goes to infinity.)

3.2 Probabilistic analysis with union bound

We now consider the probability space of choosing $2\ell + 1$ vertices from the $2n$ vertices. Recall that $2n = (2m+1)\ell + r$, with $0 \leq r \leq 2m$. Let A_i be the event that less than two neighbors of vertex i are chosen. Then for $i \in [2n]$

$$\Pr(A_i) = \frac{\binom{2n-2m-1}{2\ell}\binom{2m+1}{1} + \binom{2n-2m-1}{2\ell+1}}{\binom{2n}{2\ell+1}}.$$

When $2n$ and $2m+1$ are large and close enough, the inequality holds (e.g. $2n = 60$, $2m+1 = 57$). Experimental results shows the following.

Observation 1. *When n and m are close enough,*

$$\sum_{i \in [2n]} \Pr(A_i) < 1. \tag{3}$$

A necessary condition when Eq. (3) holds is $2n/(2m+1) < 2$.

4 Conclusion

We complete the analysis for $H_{3,N}$, and some cases for $H_{5,N}$. The results are summarized as follows.

(i) For $N = 2n$ and $2n = 3\ell + r$,

$$\gamma_{\times 2,t}(H_{3,2n}) = \begin{cases} \lceil \frac{4n}{3} \rceil + 1, & \text{if } r = 1 \text{ and } \ell \equiv 3 \bmod 4 \\ \lceil \frac{4n}{3} \rceil, & \text{otherwise.} \end{cases}$$

(ii) For $N = 2n + 1$,

$$\gamma_{\times 2,t}(H_{3,2n+1}) = \left\lceil \frac{4n+1}{3} \right\rceil.$$

(iii) For $N = 2n$ and $2n = 5\ell + r$,

$$\gamma_{\times 2,t}(H_{5,2n}) = \begin{cases} \lceil \frac{4n}{5} \rceil + 1, & \text{if } r = 1 \text{ or } r = 2 \\ \lceil \frac{4n}{3} \rceil, & \text{otherwise.} \end{cases}$$

For an n-vertex, m-regular graph, if m is large enough, we conjecture that the double total domination number matches the lower bound of $\lceil 2n/m \rceil$. Sufficient conditions for the existence of such a graph will be conducted as a future work.

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106年度專題研究計畫成果彙整表

計畫主持人：王弘倫			計畫編號：106-2221-E-003-039-MY3		
計畫名稱：Harary圖上的 k元全支配問題					
成果項目			量化	單位	質化（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文	0	篇	
		研討會論文	2	篇	Morpion Solitare 5D#的上界探討，第 36 屆組合數學與計算理論研討會。 A Note on the Double Total Domination in Harary Graphs. 第 36 屆組合數學與計算理論研討會。
		專書	0	本	
		專書論文	0	章	
		技術報告	0	篇	
		其他	0	篇	
國外	學術性論文	期刊論文	0	篇	
		研討會論文	2	篇	Determining a social choice with respect to linear preferences, Seventh International Workshop on Computational Social Choice (COMSOC-2018) Troy, NY, USA, 25-27 June 2018. The Complexity of Packing Edge-Disjoint Paths. 14th IPEC 2019: Munich, Germany 10:1-10:16
		專書	0	本	
		專書論文	0	章	
		技術報告	0	篇	
		其他	0	篇	
參與計畫人力	本國籍	大專生	0	人次	
		碩士生	7	人次	郭之淇、吳嘉雯、吳宜哲、張皓博、鍾淇翔、梁翌菖、李松展
		博士生	0	人次	
		博士級研究人員	0	人次	
		專任人員	0	人次	
	非本國籍	大專生	0	人次	
		碩士生	0	人次	
		博士生	0	人次	
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其他成果 （無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）	德國 RWTH Aachen University 之 Prof. Peter Rossmanith 研究團隊於計畫期間來訪，與其合作發表論文於 IPEC 2019 研討會。