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期末報告

神經元在雜訊與週期影響下的動態行為

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本研究具有政策應用參考價值：■否 □是，建議提供機關
（勾選「是」者，請列舉建議可提供施政參考之業務主管機關）
本研究具影響公共利益之重大發現：□否 □是

中 華 民 國 109 年 01 月 31 日

中 文 摘 要 ： 這項研究的目的是探索一類具有遞歸神經反饋的一般微分方程。首先，我們通過分析線性化方程的相應特徵方程，研究方程零解的局部穩定性。 獲得了涉及延遲和參數的一般穩定性標準。 其次，通過選擇延遲之一作為分叉參數，我們表明該方程具有霍普夫分叉。分岔週期解的穩定性通過中心流形定理和法則形式理論確定。 最後，我們將其應用於特殊模型。

中 文 關 鍵 詞 ： 一類具有遞歸神經反饋的微分方程;霍普夫分叉

英 文 摘 要 ： The purpose of this study is to explore a class of general differential-difference equations with recurrent neural feedback. First, we investigate the local stability of the zero solution of the equations by analyzing the corresponding characteristic equation of the linearized equation. General stability criteria involving the delays and the parameters are obtained. Second, by choosing one of the delays as a bifurcation parameter, we show that the equation exhibits the Hopf bifurcation. The stability of the bifurcating periodic solutions is determined by using the center manifold theorem and the normal form theory. Finally, we applied to a special model.

英 文 關 鍵 詞 ： general differential-difference equations with recurrent neural feedback; Hopf bifurcation

1. Introduction

In the field of neuroscience [Hindmarsh & Rose, 1982, 1984; Rose & Hindmarsh, 1989; FitzHugh, 1961; Nagumo *et al.*, 1962; Hodgkin & Huxley, 1952], researchers have constructed a variety of different types of biological neural networks. Early in the absence of appropriate models, the researchers use a variety of experimental methods to understand the transmission of never electrical signals. Currently, there are a variety of studies using simulation models to predict the potential behavior of neurons. Most of these modeling processes are constructed from experimental observations. To understand whether these constructed neural networks have a richer behavior, many researchers have explored the properties of the neuronal system. The main strategy is to establish a simplified neuronal model to explore the performance of the simulation at the cell level. Therefore, it is promising that both the study of a bifurcation for a reduced model and what behaviors the model possesses.

In order to understand the potential behavior of the nervous system, it is a good strategy to make the behaviors of a reduced neuronal model clear. Herein, we introduce the history of a simple neuronal model, called the Hindmarsh-Rose (HR) model. In 1970s, Connor et. al. [Connor & Stevens, 1971; Connor *et al.*, 1977] established a model which can be thought of as an alternative statement of action potential generation. The model, which is similar to the classical four-dimensional Hodgkin-Huxley (HH) model [Hodgkin & Huxley, 1952], contains fast sodium, delayed rectifier potassium, leakage, and an extra potassium conductance, called the transient A-current. The properties of the fast sodium and delayed rectifier potassium conductances are somewhat different from those of the HH model, especially the briefer action potentials caused by faster kinetics. In 1989, by a suitable transformation of a variable, Rose and Hindmarsh [Rose & Hindmarsh, 1989] simplified the six-dimensional Connor-Stevens model to the two-dimensional Hindmarsh-Rose (HR) model. Furthermore, they extend the two-dimensional HR model to the three-dimensional HR model with an additional slow variable which describes a subthreshold both inward and outward current. With the suitable parameters, the models also can resembles the models of the repetitive firing [Hindmarsh & Rose, 1982], bursting [Hindmarsh & Rose, 1984] and thalamic neurons [Rose & Hindmarsh, 1989] with detailed ionic currents, where the feature of the repetitive firing was mainly caused by the quadratic recovery term. In a physiological level, the HR model can simulate the bursting neurons of the pond snail Lymnaea [Chay, 1985b,a; Chay & Rinzel, 1985; Chay & Keizer, 1985; Sherman *et al.*, 1988]. The HR model can be seen as a generalization of the FitzHugh-Nagumo (FN) equations, which are a polynomial model and mimic most of the behaviors of the HH model [Hodgkin & Huxley, 1952]. The main difference between FN and HR is the component of the recovery variable: the former is characterized by the linearity and the latter described by the quadratic function. The HR model with respect to one or two bifurcation parameters can reproduce dynamical behaviors, such as quiescence, spiking, irregular spiking, bursting and irregular bursting [Terman, 1991, 1992; González-Miranda, 2003, 2007; Innocentia *et al.*, 2007]. The advantages in choosing the HR model contain the following reasons: one is only two nonlinear terms for its vector field and the other is the construction of circuit syntheses [Lee *et al.*, 2007], where the only control parameter is the bias current. Therefore, it is interesting to explore the neural model with a quadratic recovery term.

Generally, the richness of the dynamical behaviors can be analyzed by the bifurcation theory. Therefore, we need to realize the correlations between the physiological signals and bifurcation behaviors. In 1948, Hodgkin [Hodgkin, 1948] suggested that the existences of two different types of neurons, Class I and Class II neurons, are depended on their frequency response characteristics when a constant current is injected into the cell body. Class I neurons go from steady state to oscillatory behavior through a saddle-node bifurcation. Additionally, for Class I neuron, it has been claimed that repetitive firing first appears with zero frequency (homoclinic bifurcation), latency may be arbitrarily long and intermediate-sized responses (in amplitude) are not possible. For Class II neurons, the spiking is initiated through a (subcritical) Hopf bifurcation. The result leads to the onset of oscillations with a well-defined, non-zero frequency and possibly with small amplitude, and the latency for firing is finite. With respect to the classification based on bifurcation theory, these two types of neurons have also been called Type I and Type II, respectively. Bifurcation methodologies enable us to reduce many biophysically accurate the HR type models to a two-dimensional system of ordinary differential equations of the form. It is noteworthy that in a plane system, class I excitability is essentially characterized by the quadratic nonlinearity. To date, many two- and three- dimensional HR models [Hindmarsh & Rose, 1982, 1984; Innocentia *et al.*, 2007; Storace *et al.*, 2008; Tsuji *et al.*, 2007] have been studied. However, most of these papers investigated the bifurcations of the HR model though computer simulations. Therefore, it is important to mathematically analyze the AH and SN bifurcations of the HR model with bifurcation theory.

In this paper, we present a complete-two bifurcation diagram. Based on the bifurcation analysis, we simulate several of behaviors for 2DHR model with spike and reset condition. The paper is organized as follows: In Sec. 2, a class of non-linear models with neuronal feedback is introduced. In Sec. 3, we introduce direction and stability of the Hopf bifurcation. In Sec. 4, we apply the result in Sec. 3 to the specialized model.

2. Bifurcation analysis of a class of non-linear models with neuronal feedback

2.1. *The general class of non-linear models with neuronal feedback*

In this paper, we are interested in a kind of general differential-difference equations modeling recurrent neural feedback are as follows:

$$\dot{x}(t) = c\left[f\left(x(t), x(t-\tau)\right) - y(t) + I\right], \tag{1}$$

$$\dot{y}(t) = \left[g(x(t)) - by(t) + a\right]/c, \tag{2}$$

where x and y denote the cell membrane potential and a recovery variable, respectively, and a, b, c, d, τ and k_1 are positive constants. The parameter c represents the time scale. The parameter k_2 is positive for excitatory feedback and negative for inhibitory feedback, with the strength of the feedback given by the magnitude of k_2. Because the current is ionic, we expect that its magnitude will be approximately proportional to the difference between x and the "resting potential" v_0, which is similar to the assumption in [Plant, 1981]. The parameter I denotes the membrane current or an external stimulus. If the value of I is increased and the other parameters are unchanged, the cubic function is shifted up. However, if the value of the parameter a is reduced and the other parameter values are unchanged, the quadratic function is shifted down. Hence, the effect of I is reflected though parameter a.

For instance when f is a polynomial of degree three, we obtain a FitzHugh-Nagumo model, when F is a polynomial of degree two the Izhikevich neuron model, and when F is an exponential function the Brette-Gerstner model. However, in contrast with continuous models like the FitzHugh-Nagumo model, the two latter cases diverge when spiking, and an external reset mechanism is used after a spike is emitted.

In this paper, we want this class of models to have common properties with the neuron models. To this purpose, let us make some assumptions on the functions f and g.

Assumption (A1). The function f is at least three times continuously differentiable. (cubic-like function)

Assumption (A2). The function g is at least a unimodal function. (bimodal, multimodal)

The idea of the assumptions is from Andronov-Hopf bifurcation in Scholarpedia if we aim to extend the condition to more general model.

2.2. Equilibrium points of the system with $\tau = 0$

To understand the qualitative behavior of the dynamical system defined by Eqs. (1) and (2), we begin by studying the fixed points and analyze their stability with $\tau = 0$.

$$\dot{x}(t) = c\left[f\left(x(t), x(t)\right) - y(t) + I\right], \tag{3}$$
$$\dot{y}(t) = \left[g(x(t)) - by(t) + a\right]/c, \tag{4}$$

The linear stability of an equilibrium is governed by the Jacobian matrix of the system. Let (x_0, y_0) be an equilibrium which satisfies the following two equations:

$$f(x_0, x_0, d) - y_0 + I = 0 \tag{5}$$
$$g(x_0, d) - by_0 + a = 0, \tag{6}$$

where $d \in \mathbf{R}^m$ is a parameter domain. Here, we first omit d. It follows from Eqs. (5) and (6) that the two equations $bf(x_0, x_0) - g(x_0) + bI - a = 0$ and $y_0 = f(x_0, x_0) + I$. To simplify the notation, let $\bar{f}(x_0) = f(x_0, x_0)$. To show the existence of an equilibrium of the system is equivalent to the existence of zeros as follows:

$$h(x_0) := b\bar{f}(x_0) - g(x_0) + \bar{I} = 0. \tag{7}$$

where $\bar{I} = bI - a$. Notice that the role of current I as an external force is more meaningful than a. Therefore, use $\bar{I}$ instead of $\bar{a}$. Additionally, necessary conditions of the existence of a SN bifurcation with the equilibrium (x_0, y_0) are

(SN1) $b\bar{f}'(x) - g'(x) = 0$,

where $SN1$ is the determinant of the linear part.

(Cusp1) $b\bar{f}''(x) - g''(x) = 0$,

If SN1 and Cusp1 hold, the cusp bifurcation may occur. We define a cubic-like function. $\overline{f}$ is a cubic-like function, g is a convex function (linear is a degenerate convex), then there exist two critical x^c_ℓ and x^c_{r} with the conditions $SN1$ and $SN2$.

(AH1) $c^2\bar{f}'(x) = b$,

where $AH2$ is the trace of the linear part. Let x^0_ℓ and x^0_{r} be the solution of $AH2$.

Let $G_{b,d}(x) = bf(x, d_1) - g(x, d_2)$, where $d = (d_1, d_2)$. G_b has a unique "local" maximum (resp. minimum), denoted $M_L(b, d) = G_b(x^{\ell}_{SN}(b, d))$ (resp. $m_L(b, d) = G_b(x^{\mathrm{r}}_{SN}(b, d))$). Here, "$L$" is the first letter of the word, "local".

Theorem 1.
(i) If $M_\ell(b, d) + \bar{I} < 0$, then there exists no fixed point where Condition (SN1) holds.
(ii) If $M_\ell(b, d) + \bar{I} = 0$, then there exists a fixed point $(x^*_{SN}(b, d), y^*_{SN}(b, d))$ where Condition (SN1) holds. The fixed point is non-hyperbolic. It is unstable if $c^2\bar{f}'(x^*_{SN}(b, d)) > b$ (Note: $c > 0$). The case $c^2\bar{f}'(x^*_{SN}(b, d)) = b$, $M_\ell(b, d) + \bar{I} = 0$ is a degenerate case which has double zero eigenvalues.
(iii) If $M_\ell(b, d) + \bar{I} > 0$, then there exist two fixed point $x^{\ell}_0(\bar{I}, b, d)$ and $x^{\mathrm{r}}_0(\bar{I}, b, d)$ such that

$$x^{\ell}_0(\bar{I}, b, d) < x^*_{SN}(b, d) < x^{\mathrm{r}}_0(\bar{I}, b, d).$$

The fixed point $x^{\ell}_0(\bar{I}, b, d)$ is a saddle point, and the stability of the fixed point $x^{\mathrm{r}}_0(\bar{I}, b, d)$ depends on $\bar{I}$ and on the sign:

$$T(\bar{I}, b, d, c) = c^2 f'(x^{\mathrm{r}}_0(\bar{I}, b, d)) - b.$$

(a) When $c^2\bar{f}'(x^*_{SN}(b, d)) < b$,
(b) When $c^2\bar{f}'(x^*_{SN}(b, d)) > b$,

Proof.

(ii) The trace of this matrix is $c^2\bar{f}'(x^*_{SN}(b, d)) - b$. So, the fixed point $x^*_{SN}(b, d)$ has double zero eigenvalues if $c^2\bar{f}'(x^*_{SN}(b, d)) = b$. That is, it is a degenerate case.
(iii) Since $\det(x^{\ell}_0(\bar{I}, b, d)) < 0$, two real eigenvalues have different signs. The fixed point $x^{\ell}_0(\bar{I}, b, d)$ is a saddle point. Since $\det(x^{\mathrm{r}}_0(\bar{I}, b, d)) > 0$,

■

Let us introduce the multi-index notation for a function $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}$ with a domain E. Define $|\alpha| = \alpha_1 + \alpha_2$, $\alpha! = \alpha_1! \times \alpha_2!$ with nonnegative integers α_1, α_2. Then the higher order partial derivatives are as follows:

$$f_\alpha(x_1, x_2) = \frac{\partial^{|\alpha|} f(x_1, x_2)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}. \tag{8}$$

To simplify notations, we make the partial derivative (8) [resp. the derivative (8)] at the point (x_0, x_0) [resp. the point (x_0)] as the notation f_α [resp. g_α] instead of $f_\alpha(x_0, x_0)$ [resp. $g_\alpha(x_0)$]. For example, f_{01} [resp. f_{10} and g_1] is the first-order partial derivative of the function f with respect to x_2 [resp. x_1 and x_1] at the point (x_0, x_0) [resp. (x_0, x_0) and x_0].

2.3. *Characteristic equation*

Let us consider the associated characteristic equation of Eqs. (1) and (2) as follows:

$$\det \begin{pmatrix} cf_{10} + cf_{01}e^{-\lambda\tau} - \lambda & -c \\ g_1/c & -b/c - \lambda \end{pmatrix} = 0. \tag{9}$$

Eq. (9) can be expressed as

$$F(\lambda) + G(\lambda)e^{-\lambda\tau} = 0, \tag{10}$$

where

$$F(\lambda) = \lambda^2 + a_1\lambda + a_0, \tag{11}$$

$$G(\lambda) = b_1\lambda + b_0, \tag{12}$$

where $a_1 = b/c - cf_{10}$, $a_0 = g_1 - bf_{10}$, $b_1 = -cf_{01}$, $b_0 = -bf_{01}$. The characteristic roots are purely imaginary if the parameters satisfy one of the following two conditions (C_1) and (C_2):

(C_1) $a_0^2 < b_0^2$,
(C_2) $a_0^2 > b_0^2$, $b_1^2 - a_1^2 + 2a_0 > 0$, $(b_1^2 - a_1^2 + 2a_0)^2 > 4(a_0^2 - b_0^2)$.

Applying the lemma in [Cooke & Grossman, 1982], we obtain the following results for Eq. (10): if Condition (C_1) holds and $\tau = \tau_n^+$, then Eq. (10) has a pair of purely imaginary roots $\pm i\omega_+$; if Condition (C_2) holds for $\tau = \tau_n^+$ (resp. $\tau = \tau_n^-$), then Eq. (10) has a pair of imaginary roots $\pm i\omega_+$ (resp. $\pm i\omega_-$); if neither (C_1) nor (C_2) and $\tau > 0$, then Eq. (10) has no purely imaginary roots, where

$$\omega_\pm^2 = \frac{1}{2}(b_1^2 - a_1^2 + 2a_0) \pm\sqrt{\frac{1}{4}(b_1^2 - a_1^2 + 2a_0)^2 - (a_0^2 - b_0^2)}, \tag{13}$$

$$\cos(\tau_n^\pm \omega_\pm) = \frac{b_0(\omega_\pm^2 - a_0) - \omega_\pm^2 a_1 b_1}{b_1^2\omega_\pm^2 + b_0^2}, \tag{14}$$

$$\sin(\tau_n^\pm \omega_\pm) = \frac{a_1 b_0 \omega_\pm - (a_0 - \omega_\pm^2) b_1 \omega_\pm}{b_1^2\omega_\pm^2 + b_0^2}, \tag{15}$$

and $n = 0, 1, \cdots$. Let $\lambda_{\ell,n}(\tau) = \alpha_{\ell,n}(\tau) + i\omega_{\ell,n}(\tau)$ with $\ell =$ "$-$"or"$+$" and $n = 0, 1, 2, \cdots$, where the root of Eq. (10) satisfies $\alpha_{-,n}(\tau_n^-) = 0$, $\omega_{-,n}(\tau_n^-) = \omega_-$ and $\alpha_{+,n}(\tau_n^+) = 0$, $\omega_{+,n}(\tau_n^+) = \omega_+$. If τ_n^+ and τ_n^- are bifurcation values, we must verify that the transversality conditions hold. In other words, we obtain the following transversality conditions

(T) $\dfrac{d\mathrm{Re}\lambda_{-,n}(\tau_n^-)}{d\tau} < 0$, $\dfrac{d\mathrm{Re}\lambda_{+,n}(\tau_n^+)}{d\tau} > 0$,

where

$$\mathrm{sign}\left(\frac{d\mathrm{Re}\lambda_{\pm,n}(\tau_n^\pm)}{d\tau}\right) = \mathrm{sign}\left(a_1^2 - b_1^2 - 2a_0 + 2\omega_\pm^2\right). \tag{16}$$

By Condition (C_1), there exists a pair of pure imaginary roots if $(g_1 - bf_{10})^2 < (bf_{01})^2$. By Condition ($C_2$), there exist two pairs of pure imaginary roots.

3. Direction and stability of the Hopf bifurcation

In this section, we examine the direction and stability of Eqs. (1) and (2) using Hopf bifurcation theory [Hale, 1977; Hassard *et al.*, 1981].

Let the phase space $C_2 := C([-\tau, 0], \mathbb{R}^2)$ be the Banach space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^2$ with the supremum norm. In the following, the superscript T denotes the transpose. For $\phi = (\phi_1, \phi_2)^T \in C_2$, the operator Π is defined as

$$\Pi\phi(\theta) = \begin{cases} d\phi(\theta)/d\theta, & \text{if } \theta \in [-\tau, 0), \\ L\phi(\theta), & \text{if } \theta = 0, \end{cases}$$

where L is defined as

$$L\phi = \int_{-\tau}^{0} [d\eta(\theta)]\phi(\theta), \tag{17}$$

and $\eta : [-\tau, 0] \to \mathbb{R}^2 \times \mathbb{R}^2$ is a real-valued function of bounded variation in $[-\tau, 0]$ satisfying

$$d\eta(\theta) = (\Pi_0\delta(\theta) + \Pi_1\delta(\theta + \tau))\, d\theta,$$

with

$$\Pi_0 = \begin{bmatrix} A & D \\ B & F \end{bmatrix}, \text{ and } \Pi_1 = \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix},$$

$A = cf_{10}$, $B = g_1/c$, $\widehat{C} = cf_{01}$, $D = -c$, and $F = -b/c$. For the nonlinear part, let R be defined as

$$R\phi = \begin{cases} 0, & \text{if } \theta \in [-\tau, 0), \\ \tilde{R}(\phi), & \text{if } \theta = 0, \end{cases}$$

with

$$\tilde{R}(\phi) = \begin{pmatrix} -c\phi_1^3(0)/3 - cx_0\phi_1^2(0) \\ \phi_1^2(0)/c \end{pmatrix}.$$

Therefore, Eqs. (1) and (2) are equivalent to the following operator equation:

$$\dot{u}_t = \Pi u_t + R u_t,$$

where $u = (x, y)^T$ and $u_t = u(t + \theta)$ for $\theta \in [-\tau, 0]$.

4. For Examples: Hindmarsh-Rose Type Model

4.1. *A Two-dimensional Hindmarsh-Rose Type Model*

Let us consider that the simplest possible modification of the systems described in [Tsuji *et al.*, 2007; Chen *et al.*, 2013] to simulate synaptic feedback is the following system of differential-difference equations:

$$\dot{x}(t) = c\left[k_1 x(t) - \frac{x^3(t)}{3} - y(t) \right. \\ \left. + k_2\left(x(t-\tau) - v_0\right) + I\right], \tag{18}$$

$$\dot{y}(t) = \left(x^2(t) + dx(t) - by(t) + a\right)/c. \tag{19}$$

where x and y denote the cell membrane potential and a recovery variable, respectively, and a, b, c, d, τ and k_1 are positive constants. The parameter c represents the time scale. The parameter k_2 is positive for excitatory feedback and negative for inhibitory feedback, with the strength of the feedback given by the magnitude of k_2. Because the current is ionic, we expect that its magnitude will be approximately proportional to the difference between x and the "resting potential" v_0, which is similar to the assumption in [Plant, 1981]. The parameter I denotes the membrane current or an external stimulus. If the value of I is increased and the other parameters are unchanged, the cubic function is shifted up. However, if the value of the parameter a is reduced and the other parameter values are unchanged, the quadratic function is shifted down. Hence, the effect of I is reflected though parameter a.

For convenience, the term $I - k_2 v_0$ is replaced with I, and it is assumed that $k_1 + k_2 = 1$ and $k_1 = k$ in Eqs. (18) and (19). System (18)-(19) can be recast in the following form.

$$\dot{x}(t) = c\left[kx(t) - \frac{x^3(t)}{3} - y(t) + (1-k)x(t-\tau) + I\right], \tag{20}$$

$$\dot{y}(t) = \left(x^2(t) + dx(t) - by(t) + a\right)/c. \tag{21}$$

For $k = 1$, the system is the same as in [Tsuji *et al.*, 2007; Chen *et al.*, 2013]. We can identify the number of equilibria and their stability and several codimension-one and codimension-two bifurcations, such as the SN, Hopf, Bautin and Bogdanov-Takens bifurcations. Let the point (x_0, y_0) be an equilibrium, and let $\overline{x}(t) = x(t) - x_0$ and $\overline{y}(t) = y(t) - y_0$. To simplify the notation, the variable $\overline{x}$ is replaced with x, and the equations can be transformed as follows:

$$\dot{x}(t) = c\left[(k - x_0^2)x(t) - y(t) + (1-k)x(t-\tau)\right] \\ -\frac{1}{3}cx^3(t) - cx_0x^2(t), \tag{22}$$

$$\dot{y}(t) = \left[(2x_0 + d)x(t) - by(t)\right]/c + x^2(t)/c. \tag{23}$$

The associated characteristic equation is

$$\det\begin{pmatrix} c(k - x_0^2) + c(1-k)e^{-\lambda\tau} - \lambda & -c \\ (2x_0 + d)/c & -b/c - \lambda \end{pmatrix} = 0, \tag{24}$$

or

$$\left[\lambda - c\left(k - x_0^2 + (1-k)e^{-\lambda\tau}\right)\right](\lambda + \frac{b}{c}) + (2x_0 + d) = 0. \tag{25}$$

Eq. (30) can be expressed as

$$F(\lambda) + G(\lambda)e^{-\lambda\tau} = 0, \tag{26}$$

where $F(\lambda) = \lambda^2 + a_1\lambda + a_0$ and $G(\lambda) = b_1\lambda + b_0$ with $a_0 = b(x_0^2 - k) + (2x_0 + d)$, $a_1 = b/c - c(k - x_0^2)$, $b_0 = b(k-1)$ and $b_1 = c(k-1)$. This characteristic equation determines the local stability of an equilibrium. That is, the equilibrium is stable if and only if all of the characteristic roots λ have negative real parts.

4.2. *For an example of the general model without Delay*

Let us consider a two-dimensional Hindmarsh-Rose (2DHR) type model of the following form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu), \tag{27}$$

where $\mu = (a, b, c, d)^T$, the dot denotes differentiation with respect to the independent variable t and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function defined by $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ with functions f and g as follows:

$$f(x, y) = c(x - \frac{x^3}{3} - y + I),$$
$$g(x, y) = (x^2 + dx - by + a)/c,$$

and the two variables x and y denote the cell membrane potential and a recovery variable, respectively. The parameters a, b, c and d are positive. The parameter c represents the time scale. The parameter I denotes the membrane current or external stimulus. Because the effect of a nonzero I can be studied by the parameter a, we can assume that $I = 0$.

Let us transform Eqs. (27) as the following equations.

$$\dot{x}(t) = c\left[(1 - x_0^2)x(t) - y(t)\right] - \frac{1}{3}cx^3(t) - cx_0x^2(t), \tag{28}$$
$$\dot{y}(t) = \left[(2x_0 + d)x(t) - by(t)\right]/c + x^2(t)/c, \tag{29}$$

where x_0 is an equilibrium of Eqs. (27). The associated characteristic equation is

$$\det\begin{pmatrix} c(1 - x_0^2) - \lambda & -c \\ (2x_0 + d)/c & -b/c - \lambda \end{pmatrix} = 0, \tag{30}$$

or

$$\left[\lambda - c\left(1 - x_0^2\right)\right](\lambda + \frac{b}{c}) + (2x_0 + d) = 0. \tag{31}$$

Let $\lambda_\pm$, $q_\pm$ be eigenvalues and eigenvectors, respectively, where

$$\lambda_\pm = \frac{V_r \pm i\sqrt{M_r}}{2c},$$
$$q_\pm = \sqrt{A_r}\left[\frac{-B_r \pm i\sqrt{M_r}}{2\left(d + 2x_0\right)}, 1\right], \text{ where } ||q_\pm|| = 1.$$

Let us give some notation for simplicity as follows.

a) $M_r = -b^2 + 2bc^2\left(x_0^2 - 1\right) - c^2\left(c^2\left(x_0^2 - 1\right)^2 - 4\left(d + 2x_0\right)\right)$,
b) $A_r = (d + 2x_0)/(c^2 + d + 2x_0)$,
c) $V_r = -b + c^2\left(1 - x_0^2\right)$,
d) $B_r = -b - c^2\left(1 - x_0^2\right)$,

where "r" means replacement. Note that $V_r + B_r = -2b$, $M_r > 0$ and $c > 0$. $M_r > 0$ is If c is negative, the behaviors of the system are the same as original systems up to time reverse.

Let us consider the adjoint eigenvector $p_\pm$ with $A^T p_\pm = \lambda_\pm p_\pm$, where A^T is the adjoint linear system and $< p_-, q_+ >= 1$. Here, $< p, q >= \bar{p}^T q$ is the inner product in $\mathbb{C}^2$

$$p_- = \left(i \frac{B_r^2 \sqrt{A_r M_r} + \sqrt{A_r} M_r^{3/2}}{4 A_r c^2 M_r}, \frac{1}{2} \left(\frac{1}{\sqrt{A_r}} + i \frac{B_r}{\sqrt{A_r M_r}} \right) \right)$$

If $V_r = 0$, the system has a pair of pure imaginary, $\pm i \frac{\sqrt{M_r}}{2c}$. Under the condition $V_r = 0$, the equilibrium must be (x_0, y_0), where $x_0 = \pm \frac{\sqrt{c^2-b}}{c}$ with $c^2 > b$ and $y_0 = x_0 - x_0^3/3$. If $c^2 = b$, then two equilibria will collapse into one equilibrium. If $c^2 < b$, there exists no equilibrium. That is, there may exist a Fold bifurcation when $c^2 = b$.

Determine the zeros of the following cubic function:

$$h(x) = \frac{b}{3} x^3 + x^2 + (d - b)x + a, \tag{32}$$

where $b \neq 0$. Let (x_0, y_0) be an equilibrium of the system with

$$\frac{b}{3} x_0^3 + x_0^2 + (d - b)x_0 + a = 0. \tag{33}$$

A necessary condition of the existence of pure imaginary is the first component of the equilibrium with $x_0^\pm = \pm\sqrt{c^2 - b}/c$. As the symbol x_0 in Eq. (33) is replaced by $\pm\sqrt{c^2 - b}/c$ in Eq. (33), the following condition

$$d^\pm = \frac{\mp 3(a+1)c^3 + b^2\sqrt{c^2-b} + bc\left(2c\sqrt{c^2-b} \pm 3\right)}{3c^2\sqrt{c^2-b}} \tag{34}$$

or

$$a^\pm = \frac{\pm b^2\sqrt{c^2-b} - 3c^2\left(c \pm d\sqrt{c^2-b}\right) + bc\left(3 \pm 2c\sqrt{c^2-b}\right)}{3c^3} \tag{35}$$

must hold. Note that Eq. (34) is equivalent to Eq. (35).

Case 1. If $x_0 = -\frac{\sqrt{c^2-b}}{c}$,
then $M_r = 4c\left(2\sqrt{c^2-b} + cd\right) - 4b^2$, $A_r = \frac{2\sqrt{c^2-b}+cd}{2\sqrt{c^2-b}+c^3+cd}$, $B_r = -2b$ and $V_r = 0$.

Case 2. If $x_0 = \frac{\sqrt{c^2-b}}{c}$,
then $M_r = -4c\left(2\sqrt{c^2-b} + cd\right) - 4b^2$, $A_r = \frac{-2\sqrt{c^2-b}+cd}{2\sqrt{c^2-b}+c^3+cd}$, $B_r = -2b$ and $V_r = 0$.

5. Conclusion

In the report, we plan to study the generalized result for a class of non-linear models with neuronal feedback and give an example to show the benefit of the study. The report gives us partial results about the classification of equilibrium and Hopf bifurcation and application in a special case.

Acknowledgments

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科技部補助專題研究計畫項下出席國際學術會議心得報告

日期：109 1 月 31

出國人員姓名	陳賢修	服務機構及職稱	國立臺灣師範大學數學系 副教授
會議時間	108 年 8 月 26 日至 108 年 8 月 31 日	會議地點	(中文) 未來大學 (英文) Future University
會議名稱	(中文) 2019 國際非線性分析會議：技術與應用 (英文) 2019 International Conference on Nonlinear Analysis: Techniques and Applications		

一、參加會議經過：

國際非線性分析會議為 2019 年 8 月 26 日至 8 月 31 日在未來大學舉辦的國際學術研討會。這次會議的目的是結合非線性分析的研究學者，就會議的通用領域和應用領域交換科學信息，並討論此類主題在理論和適用方面的最新進展。本次會議發表的題目為一般型迴饋神經網絡模型中的 generalized Hopf Bifurcation。此模型可視為神經元之連續耦合，主要建構基礎在於大腦具有可塑性。期待藉由簡單的神經元模型，架構無限的可能性。本次研究重心分在 generalized Hopf bifurcation 的分析，希望藉由研究此類 bifurcation，以了解神經網路中雙重穩定的現象，特別是第一類神經和第二類神經的研究。的此方程式加強了神經元的基本功能同時推廣到一般性，以了解在可以適用的範圍。在研究的過程中，基本的研究方法不變的情形下，我們發現有更多的適用的類型出現。同樣地，我們需要建立適當的線性空間以降低維度，輔以特殊的標準式來建立腦行動態系統解的存在參數範圍。推廣的神經模型的研究成果與技術感興趣的人越來越多，預期這部份的成果將受到注意。

二、與會心得

對此次參與國際研討會，首先，我們要感謝科技部的支持與補助。此次的會議為東亞學術交流會議，因此，對於台灣與東亞學術交流進展十分有幫助。未來大學佔地不大，位於函館近郊。這次大會，有數位參與會議的研究人員針對本次演講的內容，提供不少的建議與看法，其中，特別有人提問除了分叉之外，是否還有其他的方法可以証明極限環的存在性，可以在局部分析上，再加上全域的結果。提供我們另外的觀點。對於與會者提出的問題，在未來的研究上，我們會再進一步討論分析是否除了局部分析外，有更多的結論，同時在本模型的解釋上有更好的貢獻。

三、建議

無

四、攜回資料名稱及內容

1. 大會提供的手冊。

107年度專題研究計畫成果彙整表

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