科技部補助專題研究計畫成果報告
期末報告

加權近舒爾引理

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中 文 摘 要 ： 在這個工作，我們先注重在加權的黎曼流形上，且以m-Bakry-Émery和∞-Bakry-Émery 瑞奇張量(Ricci tensor)將我們的結果分成四個部分。其中三個已經完成，最後一個我也得到相似的近舒爾不等式，但是在不等號成立時，目前只有部分結果，還需再探討。且這篇文章已經發表。

中 文 關 鍵 詞 ： 加權黎曼流形、近舒爾引理、類愛因斯坦

英 文 摘 要 ： In this work, we focus on closed weighted Riemannian manifolds. By the condition of m-Bakry-Émery and ∞-Bakry-Émery Ricci tensor, there are four results. We have complete three results in this work. At the last one, we have the almost Schur inequality, but we only have partial results whenever the equality holds.

英 文 關 鍵 詞 ： weighted Riemannian manifold, almost Schur Lemma, pseudo-Einstein

報告內容:

(一). 前言、研究目的:

In this work, we prove almost Schur Lemma on closed smooth metric measure spaces, which implies the results of X. Cheng [6] and De Lellis-Topping [11] whenever the weighted function f is constant.

(二). 文獻探討及研究方法:

In 2012, De Lellis and C. Topping [11] proved an almost Schur Lemma, that is, if a closed Riemannian manifold has nonnegative Ricci curvature, they showed an almost Schur inequality involves scalar curvature and Ricci curvature:

$$\int_M \left(R-\overline{R}\right)^2 dv \leq \frac{4n(n-1)}{(n-2)^2}\int_M \left|Ric-\frac{R}{n}g\right|^2 dv.$$

In particular, the equality holds if and only if this manifold is Einstein and constant scalar curvature.

Later, in [9], Y. Ge and G. Wang proved the almost Schur Lemma without the condition of nonnegative Ricci curvature in 4-dimension closed Riemannian manifold, i.e. they just assume the nonnegative of scalar curvature (or see [2] for 4-dimension closed Riemannian manifold with Yamabe invariant).

In [6], X. Cheng considered closed Riemannian manifolds with negative Ricci curvature and obtained a generalization of the De Lellis-Topping's theorem (or see [7]). For more references, see [5] [4] [10] [15].

In this work, we study almost Schur Lemma on a smooth metric measure space. First of all, we recall some definitions of smooth metric measure space.

For an n-dimensional closed Riemannian manifolds (M^n, g) and a smooth function f on M. A triple (M^n, g, dv_f) is a smooth metric measure space with a weighted volume identity $dv_f = e^{-f(x)}dv$, where dv is the volume element of M with respect to the metric g. Let $(\nabla f\otimes\nabla u)_{ij} = \frac{1}{2}\left(f_{,i}u_{,j}+f_{,j}u_{,i}\right)$, and let $Hess$ be the Hessian of the metric g, we define the weighted Laplacian by the trace of

$$(Hess_f u)_{ij} \equiv (Hess u)_{ij} - (\nabla f\otimes\nabla u)_{ij},$$

i.e.

$$\Delta_f u = \Delta u - \langle\nabla f, \nabla u\rangle$$

and it is a self-adjoint operator concerning dv_f.

It is natural to consider the m-Bakry-卌 ery and ∞-Bakry-卌 ery Ricci tensor on smooth metric measure space by

$$Ric_f^m = Ric + Hessf - \frac{1}{m}\nabla f\otimes\nabla f,\ m>0,$$

and

$$Ric_f = Ric + Hessf,$$

respectively. If $Ric_f = \lambda g$ (or $Ric_f^m = \lambda g$) for some $\lambda \in \mathbb{R}$, then M is quasi-Einstein (or m-quasi-Einstein), and $Ric_f = \lambda g$ is the gradient Ricci soliton equation. If f is a constant, then M is called Einstein.

According to the classical Bochner's formula, we have a similar formula

$$\tfrac{1}{2}\Delta_f \left|\nabla u\right|^2 = \left|Hessu\right|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + Ric_f\left(\nabla u, \nabla u\right)$$

for $u \in C^3\left(M\right)$ on M. Hence, there are many results may be extended from Riemannian manifolds to smooth metric measure spaces. We refer the reader to, for example [3] [8] [13] [14] [15] [16] [17] [18] for further references.

In fact, if R is the scalar curvature of M with respect to the metric g,

$$N_f^m \equiv \left(R + \tfrac{2(m-1)}{m}\Delta f - \tfrac{m-1}{m}\left|\nabla f\right|^2\right) e^{-\frac{2}{m}}$$

and

$$\overline{N}_f^m = \tfrac{\int_M N_f^m dv_f}{\int_M dv_f},$$

J.-Y. Wu [15] generalized De Lellis-Topping's result as follows:

Let $\left(M^n, g, e^{-f}dv\right)$ be a closed smooth metric measure space. For any positive number $m \neq 2$, if

$$Ric_f^m \geq \tfrac{\left|\nabla f\right|^2}{m} g,$$

then

$$\int_M \left(N_f^m - \overline{N}_f^m\right)^2 e^{-f}dv \leq \tfrac{4(m+1)(m-2)}{m^3}\int_M \left|Ric_f^m + \tfrac{trRic_f^m}{m-2}g\right|^2 e^{-\frac{m+4}{m}f}dv.$$

Moreover equality holds if and only if

$$Ric_f^m + \tfrac{trRic_f^m}{m-2}g = 0.$$

In this work, for convenience, unless otherwise specified, we give some notations as follows:

$$\begin{cases} R_f = R + \Delta f, & V_f(M) = \int_M dv_f, \\ \overline{R} = \frac{\int_M R dv_f}{V_f(M)}, & \overline{R_f} = \frac{\int_M R_f dv_f}{V_f(M)}, \\ \mathring{Ric} = Ric - \frac{R}{n}g, & \mathring{Ric}_f = Ric_f - \frac{R_f}{n}g. \end{cases}$$

(三). 結果與討論:

Now we state our results:

Theorem 1 *Let* (M^n, g, dv_f), $n > 2$, *be a closed smooth metric measure space. If*

$$Ric_f \geq (\Delta f - (n-1) K) g,$$

then

$$\left\| R_f - \overline{R_f} \right\|_{L^2} \leq \tfrac{2n\sqrt{A}}{n-2} \left\| \overset{\circ}{Ric}_f - Hessf \right\|_{L^2} + \|\Delta f\|_{L^2}, \tag{1}$$

where $\|\cdot\|_{L^2}^2 = \int_M |\cdot|^2 dv_f$,

$$A = \tfrac{n-1}{n} + \tfrac{1}{\lambda_1} (n-1) K,$$

and λ_1 *is the first positive eigenvalue of the weighted Laplacian* Δ_f. *Moreover, equality holds if and only if* M *is Einstein and constant scalar curvature with respect to metric* g.

Theorem 2 *Let* (M^n, g, dv_f), $n > 2$, *be a closed smooth metric measure space. If*

$$Ric_f^m \geq \left(\tfrac{1}{m} |\nabla f|^2 - (n-1) K \right) g$$

for any positive constant m, *then*

$$\int_M \left(R - \overline{R} \right)^2 dv_f \leq \tfrac{4n^2 A}{(n-2)^2} \int_M \left| \overset{\circ}{Ric} \right|^2 dv_f. \tag{2}$$

where

$$A = \tfrac{n-1}{n} + \tfrac{m}{2} + \tfrac{(m+2)(n-1)}{2\lambda_1} K.$$

Moreover, equality holds if and only if M *is Einstein and constant scalar curvature with respect to metric* g.

Theorem 3 *Let* (M^n, g, dv_f), $n > 2$, *be a closed smooth metric measure space. If*

$$Ric_f \geq (\Delta f - (n-1) K) g,$$

then

$$\int_M \left(R-\overline{R}\right)^2 dv_f \le \frac{4n^2A}{(n-2)^2}\int_M \left|\overset{\circ}{Ric}\right|^2 dv_f \tag{3}$$

where

$$A=\frac{n-1}{n}+\frac{1}{\lambda_1}(n-1)K.$$

Remark 4 *We note that our results are sharp since the constant $\frac{4n^2A}{(n-2)^2}$ is the same as the results in [6] (for $K>0$) and [11] (for $K=0$). So Theorem 1 implies the results of [11] and [6] whenever we select f is a constant.*

Remark 5 *In Theorem 1, 2, 3, we may select f such that $\int f dv_f = 0$ since (1), (2) and (3) are valid whenever we replace f by $f-\bar{f}$, where $\bar{f}=\frac{\int_M f dv_f}{V_f(M)}$.*

Remark 6 *In Theorem 3, if equality of (3) holds, the issue "M is trivial Einstein and constant scalar curvature" is still an open problem for us, but we prove some partial results in the section 3.*

Proof of Theorem 1:

Proof. Assume R is the nontrivial scalar curvature on M with respect to metric g, and $R_f = R+\Delta f$.

According to Sobolev embedding theorem and calculus variation, there exists a nontrivial solution $u: M\to R$ of

$$\begin{cases} \Delta_f u = R_f-\overline{R_f} \\ \int_M u dv_f = 0, \end{cases} \tag{4}$$

where

$$\overline{R_f}=\frac{\int_M R_f dv_f}{V_f(M)}$$

with $V_f(M)=\int_M dv_f$.

Since the second Bianchi identity div $Ric=\frac{1}{2}\nabla R$ implies

$$\begin{aligned}
(\text{div}\, Ric_f)_j &= (\text{div}\, Ric)_j+(\text{div}\, Hessf)_j \\
&= \nabla_i R_{ij}+(\text{div}\, Hessf)_j \\
&= \tfrac{1}{2}R_{,j}+(\text{div}\, Hessf)_j \\
&= \tfrac{1}{2}R_{f,j}-\tfrac{1}{2}(\Delta f)_{,j}+(\text{div}\, Hessf)_j\,.
\end{aligned}$$

Hence

$$\left(\operatorname{div} \overset{\circ}{Ric}_f\right)_j = \left(\operatorname{div} Ric_f\right)_j - \frac{R_{f,j}}{n} = \frac{n-2}{2n} R_{f,j} - \frac{1}{2}\left(\Delta f\right)_{,j} + \left(\operatorname{div} Hessf\right)_j ,$$

i.e.

$$\operatorname{div} \overset{\circ}{Ric}_f = \frac{n-2}{2n} \nabla R_f - \frac{1}{2}\nabla \Delta f + \operatorname{div} Hessf, \tag{5}$$

where $\overset{\circ}{Ric}_f = Ric_f - \frac{R_f}{n} g$. Then, by using

$$\begin{aligned} \int_M \left\langle \overset{\circ}{Ric}_f, hg \right\rangle dv_f &= \int_M \left\langle Ric_f - \frac{R_f}{n} g, hg \right\rangle dv_f \\ &= \int_M \left(R_f - R_f\right) h dv_f \\ &= 0, \end{aligned}$$

one has

$$\begin{aligned} \int_M \left(R_f - \overline{R_f}\right)^2 dv_f &= \int_M \left(R_f - \overline{R_f}\right) \Delta_f u dv_f = -\int_M \left\langle \nabla R_f, \nabla u \right\rangle dv_f \\ &= \frac{-2n}{n-2} \int_M \left\langle \operatorname{div} \overset{\circ}{Ric}_f + \frac{1}{2}\nabla\Delta f - \operatorname{div} Hessf, \nabla u \right\rangle dv_f \\ &= \frac{2n}{n-2} \int_M \left\langle \overset{\circ}{Ric}_f - Hessf, Hess_f u \right\rangle + \frac{1}{2}\Delta f \Delta_f u dv_f \\ &= \frac{2n}{n-2} \int_M \left\langle \overset{\circ}{Ric}_f - Hessf, Hess_f u - hg \right\rangle + \frac{n-2}{2n}\Delta f \Delta_f u dv_f \\ &\leq \frac{2n}{n-2} \left\| \overset{\circ}{Ric}_f - Hessf \right\|_{L^2} \left\| Hess_f u - hg \right\|_{L^2} + \int_M \Delta f \Delta_f u dv_f \end{aligned} \tag{6}$$

where $\left\|\cdot\right\|_{L^2}^2 = \int_M \left|\cdot\right|^2 dv_f$, and we select

$$h = \frac{\Delta_f u}{n}. \tag{7}$$

By Bochner's formula

$$\frac{1}{2}\Delta_f \left|\nabla u\right|^2 = \left|Hessu\right|^2 + \left\langle \nabla u, \nabla \Delta_f u \right\rangle + Ric_f\left(\nabla u, \nabla u\right),$$

we have

$$\begin{aligned} & \int_M \left| Hess_f u - \frac{\Delta_f u}{n} g \right|^2 dv_f \\ = \; & \int_M \left|Hess_f u\right|^2 - \frac{\left(\Delta_f u\right)^2}{n} dv_f \\ = \; & \int_M \left|Hessu\right|^2 - 2Hessu\left(\nabla f, \nabla u\right) + \frac{\left|\nabla f\right|^2 \left|\nabla u\right|^2 + \left\langle \nabla f, \nabla u\right\rangle^2}{2} - \frac{\left(\Delta_f u\right)^2}{n} dv_f \\ \leq \; & \int_M \left(1 - \frac{1}{n}\right)\left(\Delta_f u\right)^2 - Ric_f\left(\nabla u, \nabla u\right) - \left\langle \nabla f, \nabla \left|\nabla u\right|^2 \right\rangle + \left|\nabla f\right|^2 \left|\nabla u\right|^2 dv_f \\ = \; & \int_M \left(1 - \frac{1}{n}\right)\left(\Delta_f u\right)^2 - Ric_f\left(\nabla u, \nabla u\right) + \Delta f \left|\nabla u\right|^2 dv_f \\ \leq \; & \int_M \left(1 - \frac{1}{n}\right)\left(\Delta_f u\right)^2 + \left(n-1\right) K \left|\nabla u\right|^2 dv_f \end{aligned} \tag{8}$$

whenever $Ric_f \geq \left(\Delta f - \left(n-1\right) K\right) g$.

Since the first positive eigenvalue λ_1 of weighted Laplacian on M is characterized by

$$\lambda_1 = \inf\left\{ \frac{\int_M \left|\nabla \varphi\right|^2 dv_f}{\int_M \varphi^2 dv_f} \mid \varphi \text{ is nontrivial and } \int_M \varphi dv_f = 0 \right\},$$

we get

$$\begin{aligned}\int_M |\nabla u|^2 dv_f &= -\int_M u\Delta_f u dv_f = -\int_M u\left(R_f - \overline{R_f}\right) dv_f \le \|u\|_{L^2} \left\|R_f - \overline{R_f}\right\|_{L^2} \\ &\le \lambda_1^{-1/2} ||\nabla u||_{L^2} \left\|R_f - \overline{R_f}\right\|_{L^2},\end{aligned}$$

for which it holds that

$$\lambda_1 \int_M |\nabla u|^2 dv_f \le \left\|R_f - \overline{R_f}\right\|_{L^2}^2 \text{ and } \lambda_1^2 \int_M u^2 dv_f \le \left\|R_f - \overline{R_f}\right\|_{L^2}^2. \tag{9}$$

So (8) becomes

$$\int_M \left|Hess_f u - \frac{\Delta_f u}{n} g\right|^2 dv_f \le A \left\|R_f - \overline{R_f}\right\|_{L^2}^2, \tag{10}$$

where

$$A = \frac{n-1}{n} + \frac{1}{\lambda_1}(n-1)K,$$

and we may rewrite (6) as

$$\begin{aligned}&\int_M \left(R_f - \overline{R_f}\right)^2 dv_f \\ \le\ & \frac{2n}{n-2} \left\|\mathring{Ric}_f - Hessf\right\|_{L^2} \left\|Hess_f u - \frac{\Delta_f u}{n} g\right\|_{L^2} + \int_M \Delta f \Delta_f u dv_f \\ \le\ & \frac{2n\sqrt{A}}{n-2} \left\|R_f - \overline{R_f}\right\|_{L^2} \left\|\mathring{Ric}_f - Hessf\right\|_{L^2} + \int_M \Delta f \left(R_f - \overline{R_f}\right) dv_f \\ \le\ & \frac{2n\sqrt{A}}{n-2} \left\|R_f - \overline{R_f}\right\|_{L^2} \left\|\mathring{Ric}_f - Hessf\right\|_{L^2} + \left\|R_f - \overline{R_f}\right\|_{L^2} \|\Delta f\|_{L^2},\end{aligned}$$

i.e.

$$\left\|R_f - \overline{R_f}\right\|_{L^2} \le \frac{2n\sqrt{A}}{n-2} \left\|\mathring{Ric}_f - Hessf\right\|_{L^2} + \|\Delta f\|_{L^2}. \tag{11}$$

If " $=$ " of (11) holds, then we have the following properties:

(i) $Ric_f(\nabla u, \cdot) = (\Delta f - (n-1)K)\, g(\nabla u, \cdot)$,

(ii) $\mu_1(\mathring{Ric}_f - Hessf) = Hess_f u - \frac{\Delta_f u}{n} g$, where μ_1 is a non-zero constant,

(iii) $R_f - \overline{R_f} = -\lambda_1 u = \mu_2 \Delta f$, where μ_2 is a non-zero constant,

(iv) $f = \alpha u$, where α is constant (since $\int_M f dv_f = 0$).

Since (iii) and (iv) imply $\Delta_f f = \alpha \Delta_f u = \alpha\mu_2 \Delta f$, hence

$$(1 - \alpha\mu_2)\Delta f = |\nabla f|^2$$

infers that f must be zero on M. Then theorem follows by the results as in [6] and [11]. ■

Proof of Theorem 2 :

In the following, we show almost Schur lemma under the condition of m-Bakry-卌 ery Ricci tensor which is similar to the work of [15].

Proof. Now we consider the nontrivial solution $u : M \to \mathbb{R}$ of

$$\begin{cases} \Delta_f u = R - \overline{R} \\ \int_M u dv_f = 0 \end{cases} \tag{12}$$

where

$$\overline{R} = \frac{\int_M R dv_f}{V_f(M)}.$$

Since the second Bianchi identity div $Ric = \frac{1}{2}\nabla R$ implies

$$\text{div}\, \overset{\circ}{Ric} = \frac{n-2}{2n} \nabla R$$

where $(\text{div}\, Ric)_j = \nabla_i R_{ij}$ and $\overset{\circ}{Ric} = Ric - \frac{R}{n} g$.

Then, one has

$$\begin{aligned} \int_M \left(R - \overline{R}\right)^2 dv_f &= \int_M \left(R - \overline{R}\right) \Delta_f u dv_f = -\int_M \langle \nabla R, \nabla u \rangle dv_f \\ &= \frac{-2n}{n-2} \int_M \left\langle \text{div}\, \overset{\circ}{Ric}, \nabla u \right\rangle dv_f \\ &= \frac{2n}{n-2} \int_M \left\langle \overset{\circ}{Ric}, Hess_f u \right\rangle dv_f \\ &= \frac{2n}{n-2} \int_M \left\langle \overset{\circ}{Ric}, Hess_f u - hg \right\rangle dv_f \\ &\leq \frac{2n}{n-2} \left\| \overset{\circ}{Ric} \right\|_{L^2} \| Hess_f u - hg \|_{L^2} . \end{aligned} \tag{13}$$

Now we select

$$h = \frac{\Delta_f u}{n}.$$

By Bochner's formula

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |Hessu|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + Ric_f (\nabla u, \nabla u),$$

we have

$$\begin{aligned} & \int_M \left| Hess_f u - \frac{\Delta_f u}{n} g \right|^2 dv_f \\ = \; & \int_M |Hessu - \nabla f \otimes \nabla u|^2 - \frac{\left(\Delta_f u\right)^2}{n} dv_f \\ \leq \; & \int_M \left(1 + \frac{m}{2}\right) |Hessu|^2 + \left(1 + \frac{2}{m}\right) |\nabla f \otimes \nabla u|^2 - \frac{\left(\Delta_f u\right)^2}{n} dv_f \\ = \; & \int_M \left(1 - \frac{1}{n} + \frac{m}{2}\right) (\Delta_f u)^2 - \frac{m+2}{2} Ric_f (\nabla u, \nabla u) \\ & + \frac{m+2}{2m} \left(|\nabla f|^2 |\nabla u|^2 + \langle \nabla f, \nabla u \rangle^2 \right) dv_f \\ \leq \; & \int_M \left(\frac{n-1}{n} + \frac{m}{2} \right) (\Delta_f u)^2 + \frac{(m+2)(n-1)K}{2} |\nabla u|^2 dv_f, \end{aligned} \tag{14}$$

here we use $Ric_f^m \geq \left(\frac{1}{m} |\nabla f|^2 - (n-1) K \right) g$.

Hence, by the inequality of eigenvalue λ_1 (see (9)), (14) gives

$$\int_M \left| Hess_f u - \frac{\Delta_f u}{n} g \right|^2 dv_f \leq A \left\| R - \overline{R} \right\|_{L^2}^2 ,$$

and then one has

$$\left\|R-\overline{R}\right\|_{L^2}\leq\frac{2n\sqrt{A}}{n-2}\left\|\overset{\circ}{Ric}\right\|_{L^2} \tag{15}$$

where

$$A=\left(\frac{n-1}{n}+\frac{m}{2}\right)+\frac{(m+2)(n-1)}{2\lambda_1}K$$

for any constant $m>0$.

If " $=$ " holds, then $Hessu=\frac{2}{m}\nabla f\otimes\nabla u$ on M. For which it implies

$$\Delta_{\frac{2f}{m}}u=\Delta u-\frac{2}{m}\left\langle\nabla f,\nabla u\right\rangle=0, \tag{16}$$

i.e. u is a weighted harmonic function with respect to weighted measure $dv_{\frac{2f}{m}}$ on M, it infers $u=0$ on M. So Theorem 2 follows. ■

Combine Theorem 2 and Theorem 1, it is clear that one has the following property.

Corollary 7 *Let* (M^n,g,dv_f), $n>2$, *be a closed smooth metric measure space. If*

$$Ric_f^m\geq\left(\frac{1}{m}\left|\nabla f\right|^2-(n-1)K\right)Kg$$

for any positive constant m, *then*

$$\left\|R_f-\overline{R_f}\right\|_{L^2}\leq\frac{2n\sqrt{A}}{n-2}\left\|\overset{\circ}{Ric}_f-Hessf\right\|_{L^2}+\left\|\Delta f\right\|_{L^2},$$

where $\left\|\cdot\right\|_{L^2}^2=\int_M(\cdot)^2\,dv_f$ *and*

$$A=\frac{n-1}{n}+\frac{m}{2}+\frac{(m+2)(n-1)}{2\lambda_1}K.$$

Moreover, equality holds if and only if M *is Einstein and constant scalar curvature with respect to metric* g.

Proof of Theorem 3 :

As the process form (12) to (15) in the proof of Theorem 2, but we replace (14) by the following formula,

$$\begin{aligned}
&\int_M\left|Hess_fu-\frac{\Delta_fu}{n}g\right|^2dv_f\\
=\ &\int_M\left|Hess_fu\right|^2-\frac{\left(\Delta_fu\right)^2}{n}dv_f\\
=\ &\int_M\left|Hessu\right|^2-2Hessu\left(\nabla f,\nabla u\right)+\frac{|\nabla f|^2|\nabla u|^2+\langle\nabla f,\nabla u\rangle^2}{2}-\frac{\left(\Delta_fu\right)^2}{n}dv_f\\
\leq\ &\int_M\left(1-\frac{1}{n}\right)\left(\Delta_fu\right)^2-Ric_f\left(\nabla u,\nabla u\right)-\left\langle\nabla f,\nabla\left|\nabla u\right|^2\right\rangle+\left|\nabla f\right|^2\left|\nabla u\right|^2dv_f\\
=\ &\int_M\left(1-\frac{1}{n}\right)\left(\Delta_fu\right)^2-Ric_f\left(\nabla u,\nabla u\right)+\Delta f\left|\nabla u\right|^2dv_f\\
\leq\ &\int_M\left(1-\frac{1}{n}\right)\left(\Delta_fu\right)^2+(n-1)K\left|\nabla u\right|^2dv_f
\end{aligned}$$

whenever $Ric_f \geq (\Delta f - (n-1)K) g$. So we obtain

$$\int_M \left| Hess_f u - \frac{\Delta_f u}{n} g \right|^2 dv_f \leq A \int_M \left(R - \overline{R} \right)^2 dv_f, \tag{17}$$

and then the inequality (3)

$$\int_M \left(R - \overline{R} \right)^2 dv_f \leq \frac{4n^2 A}{(n-2)^2} \int_M \left| \overset{\circ}{Ric} \right|^2 dv_f \tag{18}$$

holds, where

$$A = \frac{n-1}{n} + \frac{1}{\lambda_1} (n-1) K.$$

If " $=$ " of (18) holds, we have the following properties:

(i) $Ric_f (\nabla u, \cdot) = (\Delta f - (n-1) K) g (\nabla u, \cdot)$,

(ii) $\mu \overset{\circ}{Ric} = Hess_f u - \frac{\Delta_f u}{n} g$, where μ is a non-zero constant,

(iii) $R - \overline{R} = -\lambda_1 u$,

(iv) $f = \alpha u$, where α is a constant.

In the following, we prove that if " $=$ " of (18) holds, and under the condition $\alpha \leq \frac{1}{n-1}$, then M is Einstein and constant scalar curvature with respect to metric g, but it is still an open problem whenever $\alpha > \frac{1}{n-1}$.

Remark 8 *If $\alpha = 0$, then theorem follows by [6] (or [11] for $K = 0$). So we assume $\alpha \neq 0$.*

Lemma 9 *μ must satisfy $\mu = \frac{2nA}{n-2} = \frac{2n}{n-2} \left(\frac{n-1}{n} + \frac{1}{\lambda_1} (n-1) K \right)$, or M is trivial Einstein and constant scalar curvature with respect to metric g.*

Proof. By (ii), (17) and (18), we have

$$\begin{aligned} \mu^2 \int_M \left| \overset{\circ}{Ric} \right|^2 dv_f &= \int_M \left| Hess_f u - \frac{\Delta_f u}{n} g \right|^2 dv_f = A \left\| R - \overline{R} \right\|_{L^2}^2 \\ &= \frac{4n^2 A^2}{(n-2)^2} \int_M \left| \overset{\circ}{Ric} \right|^2 dv_f \end{aligned}$$

which gives

$$\left(\mu^2 - \frac{4n^2 A^2}{(n-2)^2} \right) \int_M \left| \overset{\circ}{Ric} \right|^2 dv_f = 0.$$

It is clear that if $\mu \neq \frac{2nA}{n-2}$, then $\overset{\circ}{Ric} = 0$ and M is trivial Einstein and constant scalar curvature with respect to metric g. Hence $\mu = \frac{2nA}{n-2}$. ∎

By (i) and (iv),

$$R_{ij}u_{,i}+\alpha u_{,ij}u_{,i}-\alpha u_{,j}\Delta u+(n-1)\,Ku_{,j}=0$$

implies

$$R_{ij,j}u_{,i}+\alpha u_{,ijj}u_{,i}-\alpha\left(\Delta u\right)_{,j}u_{,j}+R_{ij}u_{,ij}+\alpha u_{,ij}^{2}-\alpha\left(\Delta u\right)^{2}+(n-1)\,K\Delta u=0. \tag{19}$$

And (ii) gives

$$\begin{array}{rcl}\mu R_{ij}u_{,ij} & = & \left\langle \frac{\mu R}{n}g+Hess_{f}u-\frac{\Delta_{f}u}{n}g,Hessu\right\rangle \\ & = & \frac{\mu R}{n}\Delta u+\left|Hessu\right|^{2}-\alpha Hessu\left(\nabla u,\nabla u\right)-\frac{\Delta u-\alpha\left|\nabla u\right|^{2}}{n}\Delta u.\end{array} \tag{20}$$

Now let $p\in M$ be the minimal point p of u, i.e. $u\left(p\right)=\inf_{M}u$.
Then (19) and (20) can be rewritten as

$$\left\{\begin{array}{rcl}R_{ij}u_{,ij} & = & \alpha\left(\Delta u\right)^{2}-\alpha\left|Hessu\right|^{2}-(n-1)\,K\Delta u,\\ \mu R_{ij}u_{,ij} & = & \frac{\mu R}{n}\Delta u+\left|Hessu\right|^{2}-\frac{1}{n}\left(\Delta u\right)^{2},\end{array}\right. \tag{21}$$

at p. For which we have

$$\begin{array}{rcl}0 & = & \frac{\mu R}{n}\Delta u+(1+\alpha\mu)\left|Hessu\right|^{2}-\left(\frac{1}{n}+\alpha\mu\right)\left(\Delta u\right)^{2}+(n-1)\,\mu K\Delta u\\ & = & (1+\alpha\mu)\left|Hessu\right|^{2}-\frac{1+\alpha\mu}{n}\left(\Delta u\right)^{2}+\frac{\mu}{n}\left(R-(n-1)\,\alpha\Delta u+n\,(n-1)\,K\right)\Delta u\end{array} \tag{22}$$

at p.

Since

$$\begin{array}{rcl}R-(n-1)\,\alpha\Delta u+n\,(n-1)\,K & = & \overline{R}+\Delta u-(n-1)\,\alpha\Delta u+n\,(n-1)\,K\\ & = & (1-(n-1)\,\alpha)\,\Delta u+\overline{R}+n\,(n-1)\,K,\end{array}$$

so (22) can be rewritten as

$$\begin{array}{rl} & (1+\alpha\mu)\left(\left|Hessu\right|^{2}-\frac{1}{n}\left(\Delta u\right)^{2}\right)+\frac{\mu}{n}\left(1-(n-1)\,\alpha\right)\left(\Delta u\right)^{2}\\ & +\frac{\mu}{n}\left(\overline{R}+n\,(n-1)\,K\right)\Delta u\\ = & 0\text{ at }p.\end{array} \tag{23}$$

Besides, due to curvature assumption

$$Ric+\alpha Hessu\geq\left(\alpha\Delta u-(n-1)\,K\right)g, \tag{24}$$

one has

$$R\geq\alpha\,(n-1)\,\Delta u-n\,(n-1)\,K, \tag{25}$$

and it gives

$$\begin{array}{rcl}\overline{R} & \geq & \frac{\alpha(n-1)}{V_{f}(M)}\int_{M}\Delta u dv_{f}-n\,(n-1)\,K\\ & = & \frac{\alpha^{2}(n-1)}{V_{f}(M)}\int_{M}\left|\nabla u\right|^{2}dv_{f}-n\,(n-1)\,K\\ & > & -n\,(n-1)\,K\text{ for all }\alpha\neq0,\end{array} \tag{26}$$

here we use

$$\Delta u = \Delta_f u + \alpha \left|\nabla u\right|^2 .$$

If $\frac{-1}{\mu} \leq \alpha \leq \frac{1}{n-1}$, then each term in the left hand side of (23) must be nonnegative at p, so $\Delta u\left(p\right) = 0$, which implies $R\left(p\right) = \sup_M R = \overline{R}$. Hence M is Einstein and constant scalar curvature with respect to metric g.

If $\alpha \leq -\frac{1}{\mu}$, at p, we rewrite (23) as

$$\left(1+\alpha\mu\right)\left(\left|Hessu\right|^2 - \left(\Delta u\right)^2\right) + \tfrac{(n-1)+\mu}{n}\left(\Delta u\right)^2 + \tfrac{\mu}{n}\left(\overline{R} + n\left(n-1\right)K\right)\Delta u = 0. \tag{27}$$

We note that, at p, the $n \times n$ matrix $Hessu$ must be semi-positive, then $\left|Hessu\right|^2 \leq \left(\Delta u\right)^2$ at p, and equality holds if only if the rank of $Hessu\left(p\right)$ less than 2. For which each term in the left hand side of (27) must be nonnegative. So $\Delta u\left(p\right) = R\left(p\right) - \overline{R} = 0$, then M is Einstein and constant scalar curvature with respect to metric g.

(四). 參考文獻:

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科技部補助專題研究計畫執行出國參訪及考察心得報告

日期： 108 年 10 月 29 日

計畫編號	MOST 107-2115-M-003-007 -		
計畫名稱	加權近舒爾引理		
出國人員姓名	陳瑞堂	服務機構及職稱	師大數學系副教授
出國時間	108 年 6 月 30 日至 108 年 7 月 13 日	出國地點	信陽師範大學

參訪及考察過程:

先至信陽師範大學訪問韓英波教授及參加幾何國際會議，之後到武漢大學訪問羅勇教授。

107年度專題研究計畫成果彙整表

計畫主持人：陳瑞堂				計畫編號：107-2115-M-003-007-		
計畫名稱：加權近舒爾引理						
成果項目				量化	單位	質化（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文		1	篇	論文已發表
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		專書		0	本	
		專書論文		0	章	
		技術報告		0	篇	
		其他		0	篇	
	智慧財產權及成果	專利權	發明專利 申請中	0	件	
			發明專利 已獲得	0		
			新型/設計專利	0		
		商標權		0		
		營業秘密		0		
		積體電路電路布局權		0		
		著作權		0		
		品種權		0		
		其他		0		
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		收入		0	千元	
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		研討會論文		0		
		專書		0	本	
		專書論文		0	章	
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	智慧財產權及成果	專利權	發明專利 申請中	0	件	
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（無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）</td><td colspan="3"></td></tr>
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科技部補助專題研究計畫成果自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現（簡要敘述成果是否具有政策應用參考價值及具影響公共利益之重大發現）或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以100字為限）

   　　□實驗失敗

   　　□因故實驗中斷

   　　□其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形（請於其他欄註明專利及技轉之證號、合約、申請及洽談等詳細資訊）

   論文：■已發表　□未發表之文稿　□撰寫中　□無

   專利：□已獲得　□申請中　■無

   技轉：□已技轉　□洽談中　■無

   其他：（以200字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性，以500字為限）

   我們推展了近舒爾引理到加權的流形，結果和一般流形相似且包括一般流形的情形，這是很不錯的結果。

4. 主要發現

   本研究具有政策應用參考價值：■否　□是，建議提供機關

   （勾選「是」者，請列舉建議可提供施政參考之業務主管機關）

   本研究具影響公共利益之重大發現：□否　□是

   說明：（以150字為限）