科技部補助專題研究計畫成果報告
期末報告

座落在平面凸區域上的最小網絡之組態

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報 告 附 件 ： 移地研究心得報告

本研究具有政策應用參考價值：■否 □是，建議提供機關
（勾選「是」者，請列舉建議可提供施政參考之業務主管機關）
本研究具影響公共利益之重大發現：□否 □是

中 華 民 國 109 年 01 月 29 日

中 文 摘 要 ： 在這個計畫中，我們對曲線網絡的結構感到興趣，而且該曲線網絡的彈性能量是處於通過梯度流達到的平衡狀態。 我們從較簡單的例子開始研究，其中包括類似三腳架網絡和樹型網絡。 我們的研究手法是通過在結點/邊界點設置某些邊界條件，而且將網絡視為在結點處聯接的曲線，然後我們考慮網絡的幾何梯度流。 實際上，在結點處的各種‘‘邊界條件’’為幾何梯度流設立了不同類型的解題困難度。

在各種邊界條件中，最困難的是所謂的約束型(clamped)邊界條件，即端點是固定的，任何端點處的切線指標是指定的（恆定）單位向量。我們的主要結果列在報告中說明。

中 文 關 鍵 詞 ： 二階拋物方程，約束型(clamped)邊界條件，曲線的Willmore泛函

英 文 摘 要 ： In this project, we are interested in the configurations of curve networks, whose elastic energy are at equilibria achieved by gradient flows. Simpler cases to start the study include tripod-like networks and tree-type networks. Our approach is to view networks as unions of curves joint at junction points by setting up certain boundary conditions at junction/boundary points, and then consider geometric gradient flow of networks. In fact , various of ‘‘boundary conditions" at the junction points provide different type of difficulty for the geometric gradient flow.

Among various boundary conditions, the most difficult one is the so-called clamped boundary condition, namely the end point is fixed and the tangent indicatrix at any end point is a prescribed (constant) unit vector.

The main result in this case is stated in our report.

英 文 關 鍵 詞 ： second-order parabolic equation, clamped boundary condition, Willmore functional of curves

Scientific Report

Supported by Ministry of Sciences and Technology

The configurations of minimal networks in planar convex regions
座落在平面凸區域上的最小網絡之組態

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Keywords: second-order parabolic equation, clamped boundary condition, Willmore functional of curves

Abstract

In this project, we are interested in the configurations of curve networks, whose elastic energy are at equilibria achieved by gradient flows. Simpler cases to start the study include tripod-like networks and tree-type networks. Our approach is to view networks as unions of curves joint at junction points by setting up certain boundary conditions at junction/boundary points, and then consider geometric gradient flow of networks. In fact , various of "boundary conditions" at the junction points provide different type of difficulty for the geometric gradient flow.

Among various boundary conditions, the most difficult one is the so-called clamped boundary condition, namely the end point is fixed and the tangent indicatrix at any end point is a prescribed (constant) unit vector. The main result in this case is stated as follows. *Given any C^2-smooth initial open curves in $\mathbb{R}^2$, there exists an evolution decreasing the elastic energy of inextensible curves in an efficient manner and keeping the positions and tangential directions at boundary fixed for arbitrary long time. Moreover, the asymptotic limits of a convergent subsequence are inextensible elasticae.*

In this report, we only focus on the case of elastic flow with clamped ends. Although we already obtain certain progress in the cases of networks, we leave the details to the future project report and journal articles. Instead, here we only report the improved argument we wrote in previous report, which allows us to extend the techniques to elastic networks easier.

1 Research Report（報告內容）

In this project, we are interested in the configurations of curve networks, whose elastic energy are at equilibria achieved by gradient flows. Simpler cases to start the study include tripod-like networks and tree-type networks. Our approach is to view networks as unions of curves joint at junction points by setting up certain boundary conditions at junction/boundary points, and then consider geometric gradient flow of networks. In fact , various of “boundary conditions” at the junction points provide different type of difficulty for the geometric gradient flow. Among various boundary conditions, the most difficult one is the so-called clamped boundary condition, namely the end point is fixed and the tangent indicatrix at any end point is a prescribed (constant) unit vector. However, for the case of fourth-order elastic flow, we still focus on the case close to that of hinged ends for the sake of simplicity. We would investigate more complicated cases later.

Two types of elastic flows are studied in our project. We list them in two subsections below.

1.1 the second-order elastic flow

In this subsection, we only focus on the case of elastic flow with clamped ends. Although we already obtain certain progress in the cases of networks, we leave the details to the future project report and journal articles. Instead, here we only report the improved argument we wrote in previous report, which allows us to extend the techniques to elastic networks easier.

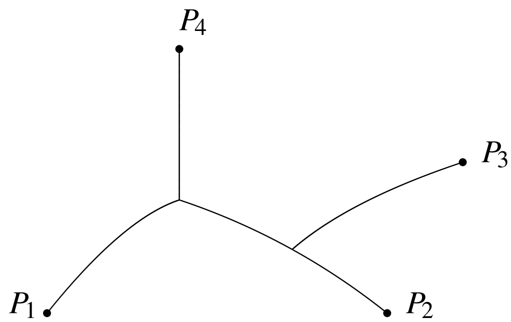

Figure 1: The network model

Geometric flows of elastic curves have been studied in various settings, for example, the type of fourth-order parabolic PDE of *closed* curves with a positive stretching coefficient [5], [6], [10], [12]; the 4th-order parabolic PDE of *open* curves with a positive stretching coefficient and clamped ends [7]; the 4th-order parabolic PDE of *open* curves of Willmore-Helfrich type with a positive stretching coefficient and hinged ends [4]; the 4th-order parabolic PDE of *open* curves with fixed total length and with hinged or clamped ends [2], [3]; the 4th-order parabolic PDE of *open* and *inextensible* plane curves with hinged ends or infinite length [9]. Among these geometric flows, various boundary conditions create different level of difficulty in deriving long-time smooth solutions.

We investigate in details the *second-order* geometric flow for elastic curves in $\mathbb{R}^2$ with *clamped* ends by considering the gradient flow of the tangent indicatrices of curves, decreasing the elastic (or more precisely bending) energy, $\int_I |\partial_s \vec{T}|^2 ds$, for example as shown in Figure 1. Namely, for $t_0 > 0$ and $I = (0, L)$, let $\vec{T} : \bar{I} \times [0, t_0) \to \mathbb{S}^1(1) \subset \mathbb{R}^2$ be the tangent indicatrices of a continuous family of plane curves and fulfill

$$\partial_t \vec{T} = \nabla_s^2 \vec{T} - \langle \vec{\lambda}, \vec{n} \rangle \vec{n}, \qquad \text{in } I \times (0, t_0), \tag{1}$$

$$\vec{T} = \text{ prescribed unit constant vectors (or } \partial_t \vec{T} = 0 \text{),} \qquad \text{on } \partial I \times (0, t_0), \tag{2}$$

$$\vec{T} = \vec{T}_0, \qquad \text{on } \bar{I} \times \{t = 0\}, \tag{3}$$

where ∇_s denotes the covariant differentiation defined in (6), $\vec{n}$ denotes the unit vector in $\mathbb{R}^2$ derived from the counterclockwise rotation of $\vec{T}$ by 90 degrees (representing the unit normal vectors of plane curves), and $\vec{\lambda} = \vec{\lambda}(t)$ is the Lagrange multiplier defined by

$$\vec{\lambda} := \left(\int_I \nabla_s^2 \vec{T} \, ds \right) \cdot A_T^{-1} = \left(\partial_s \vec{T}_{\lfloor \partial I} + \int_I |\partial_s \vec{T}|^2 \vec{T} \, ds \right) \cdot A_T^{-1}, \tag{4}$$

$$A_T := \int_I (\vec{n})^t \vec{n} \, ds \in \mathbb{M}^{2 \times 2}. \tag{5}$$

Note that in this article, $s \in \bar{I}$ represents the arclength parameter of the family of plane curves $f(s,t) = \int_0^s \vec{T}(\sigma, t) \, d\sigma + f(0,t)$, and together with $t \in [0, t_0)$ are independent variables.

To the best our knowledge, the second-order geometric flow (1) was first investigated by Wen in [11] for the case of closed curves. We extend the study of [11] to the case of open curves with hinged ends in [8]. The solutions of (1) give an alternative way of evolving plane inextensible curves, instead of fourth-order parabolic equation. There are remarkable properties for smooth solutions of (1), e.g., convexity-preserving and non-increasing of the number of inflection points during the evolution (see [1], [11]). These geometric properties also appear in curve-shortening flow of plane curves and could be useful in applications (e.g., using nonlinear splines for interpolation of data).

Let $f : \bar{I} \to \mathbb{R}^2$ be a regular, *non-closed*, plane curve, $L > 0$ represent the total length of f, and $s \in I$ be the arclength parameter of f. Let $\vec{g} : I \to \mathbb{R}^2$ be a vector field $\vec{g}$ along f. Note that the notation $\vec{g} \in C^\infty(\bar{I})$ means that $\vec{g}$ and all of its derivatives are continuous up to the boundary ∂I. Let

$$\nabla_s \vec{g} := \partial_s \vec{g} - \langle \partial_s \vec{g}, \vec{T} \rangle \vec{T}$$

denote the covariant derivative of the vector field $\vec{g}$. Similarly, for a family of vector fields $\vec{h} : I \times (0, t_0) \to \mathbb{R}^2$, we let

$$\nabla_t \vec{h} := \partial_t \vec{h} - \langle \partial_t \vec{h}, \vec{T} \rangle \vec{T}, \quad \nabla_s \vec{h} := \partial_s \vec{h} - \langle \partial_s \vec{h}, \vec{T} \rangle \vec{T}. \tag{6}$$

Denote by $\vec{T}$ the tangent indicatrix of f and by $\vec{\kappa} = \partial_s \vec{T}$ the curvature vector of f. The bending energy of plane curves is defined by

$$\mathscr{E}[f] = \int_I \frac{1}{2} |\vec{\kappa}|^2 \, ds. \tag{7}$$

When a smooth and regular plane curve $f_0 : \bar{I} \to \mathbb{R}^2$ is a critical point of the bending energy in (7), it is said to be inextensible if its perturbation class is restricted to

$$\mathscr{D}_{f_0} = \{ f \in C^\infty(\bar{I} \times (-1,1)) : f(s,0) = f_0, |\partial_s f(s,\varepsilon)| = 1, \ \forall \, s \in \bar{I}, \forall \, \varepsilon \in (-1,1) \}. \tag{8}$$

Suppose $f(0) = p_-$, $f(L) = p_+$ and $L > |p_+ - p_-|$. The family of inextensible plane curves with fixed length L and fixed end points p_-, p_+ can be equivalently described by a family of tangent indicatrices. Namely, let

$$\mathscr{A}_L := \left\{ \vec{T} : \vec{T} \in C^\infty \left(\bar{I}, \mathbb{S}^1(1) \right) \right\}$$

and

$$\mathscr{A}_{L,0} := \left\{ \vec{T} : \vec{T} \in \mathscr{A}_L, \vec{T}_{\lfloor \partial I} = \text{fixed vectors} \right\}. \tag{9}$$

Note that if the family of tangent indicatrices $\{\vec{T}(\cdot,\varepsilon) : \varepsilon \in (-1,1)\} \subset \mathscr{A}_L$ also fulfills the constraint

$$\int_I \vec{T}(s,\varepsilon)\ ds = p_+ - p_- =: \Delta p,\ \forall \text{ fixed } \varepsilon \in (-1,1),$$

then the fundamental theorem of plane curves allows us to construct a family of inextensible plane curves with fixed length L and fixed end points p_-, p_+ from the family of tangent indicatrices.

Let the functionals $\mathscr{F}_L$ and $\mathscr{F}_{L,\Delta p}$: $\mathscr{A}_L \to \mathbb{R}$ be defined respectively by

$$\mathscr{F}_L[\vec{T}] := \int_I \frac{1}{2} |\partial_s \vec{T}|^2\ ds,$$

and

$$\mathscr{F}_{L,\Delta p}[\vec{T}] := \int_I \frac{1}{2} |\partial_s \vec{T}|^2\ ds + \vec{\lambda} \cdot \left(\int_I \vec{T}\ ds - \Delta p \right), \tag{10}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ is the $\mathbb{R}^2$-valued Lagrange multiplier and $|\triangle p| < L$ is always assumed. The first variations of $\mathscr{F}_{L,\Delta p}$ in the class $\mathscr{A}_L$ gives

$$\partial_\varepsilon \mathscr{F}_{L,\Delta p}[\vec{T}(\cdot,\varepsilon)]_{|_{\varepsilon=0}} = \langle \vec{\kappa}, (\partial_\varepsilon \vec{T})_{|\varepsilon=0} \rangle_{|\partial I} - \int_I \langle \partial_s \vec{\kappa} - \vec{\lambda}, (\partial_\varepsilon \vec{T})_{|\varepsilon=0} \rangle\ ds. \tag{11}$$

By applying the boundary condition (2), the boundary term in (11) vanishes. Since $|\vec{T}(\cdot,\varepsilon)| \equiv 1$ implies that $\langle \partial_\varepsilon \vec{T}, \vec{T} \rangle \equiv 0$, i.e. $\partial_\varepsilon \vec{T}$ is parallel to $\vec{n}$, we may rewrite (11) as

$$\partial_\varepsilon \mathscr{F}_{L,\Delta p}[\vec{T}(\cdot,\varepsilon)]_{|_{\varepsilon=0}} = -\int_I \langle \nabla_s \vec{\kappa} - \langle \vec{\lambda}, \vec{n} \rangle \vec{n}, (\partial_\varepsilon \vec{T})_{|\varepsilon=0} \rangle\ ds.$$

By applying the fundamental theorem of distribution, we obtain the Euler-Lagrange equation of $\mathscr{F}_{L,\Delta p}$,

$$\nabla_s^2 \vec{T} - \langle \vec{\lambda}, \vec{n} \rangle \vec{n} = 0. \tag{12}$$

If $\vec{T} \in \mathscr{A}_{L,0}$ satisfies (12), then $\vec{T}$ is called a critical point of $\mathscr{F}_{L,\Delta p}$.

The flow (1) considered in this article is the same as the equation in [8], except the boundary conditions. As in [8], (1) could be converted into a scalar-valued equation,

$$\partial_t \varphi = \partial_s^2 \varphi + \lambda_1 \sin \varphi - \lambda_2 \cos \varphi = \partial_s^2 \varphi - \langle \vec{\lambda}, \vec{n} \rangle,$$

by letting

$$\vec{T}=(\cos\varphi,\sin\varphi), \tag{13}$$

where $\varphi:\bar{I}\times[0,t_1)\to\mathbb{R}$. From (13) and (5), one yields

$$A_T=\begin{pmatrix} \int_I \sin^2\varphi\, ds & -\int_I \sin\varphi\cos\varphi\, ds \\ -\int_I \sin\varphi\cos\varphi\, ds & \int_I \cos^2\varphi\, ds \end{pmatrix}. \tag{14}$$

After a straightforward computation, the Lagrange multipliers defined in (4) can be equivalently written as

$$\begin{cases} \lambda_1=\lambda_1^{\varphi}=(\det A_T)^{-1}\cdot\Bigg[\Big(-(\partial_s\varphi)\sin\varphi_{\lfloor\partial I}+\int_I(\partial_s\varphi)^2\cos\varphi\, ds\Big)\cdot\Big(\int_I\cos^2\varphi\, ds\Big) \\ \qquad\qquad +\Big((\partial_s\varphi)\cos\varphi_{\lfloor\partial I}+\int_I(\partial_s\varphi)^2\sin\varphi\, ds\Big)\cdot\Big(\int_I\sin\varphi\cos\varphi\, ds\Big)\Bigg], \\ \lambda_2=\lambda_2^{\varphi}=(\det A_T)^{-1}\cdot\Bigg[\Big(-(\partial_s\varphi)\sin\varphi_{\lfloor\partial I}+\int_I(\partial_s\varphi)^2\cos\varphi\, ds\Big)\cdot\Big(\int_I\sin\varphi\cos\varphi\, ds\Big) \\ \qquad\qquad +\Big((\partial_s\varphi)\cos\varphi_{\lfloor\partial I}+\int_I(\partial_s\varphi)^2\sin\varphi\, ds\Big)\cdot\Big(\int_I\sin^2\varphi\, ds\Big)\Bigg]. \end{cases} \tag{15}$$

The short-time existence of smooth solutions for the L^2-flow (1) with hinged ends and C^1-smooth initial datum of tangent indicatrices (which give C^2-smooth initial curves) was derived in [8] by using the periodicity property of entire solutions. One couldn't simply use the same argument for the case of clamped ends. Therefore we have to develop a different manner to derive short-time smooth solutions in this project. In fact, we use an implicit formula of solutions represented by volume potentials and surface potentials, then apply fixed point theorem in suitable functional spaces to obtain C^1-smooth solutions of the tangent indicatrices, then argue by potential theory to derive $C^{2,\alpha}$-smoothness of the tangent indicatrices in short-time, i.e., for all $t\in(0,t_0)$ for some $t_0>0$. Finally, we use boot-strapping argument and PDE theory to obtain instant C^∞-smoothness for all $t\in(0,t_0)$. Notice that our approach for the short-time existence result allows us to have low regularity of initial curves.

The long-time smooth solutions for the L^2-flow (1) with hinged ends and C^1-smooth initial datum of tangent indicatrices was derived in [8] by applying the Gagliardo-Nirenberg type interpolation inequalities to the estimates of L^2-norms of high-order derivatives of curvature. The hinged boundary conditions provide good algebraic structures in applying integration by parts and the Gagliardo-Nirenberg type interpolation inequalities to the higher-order energy estimates. As a gift of the nice matches, the terms involving Lagrange multipliers $\vec{\lambda}$ would not appear as the highest-order, which is however not the case as working with the clamped boundary condition. In other words, we can't use the same trick to estimate terms associated to the boundary and terms involving the derivatives of $\vec{\lambda}$, in the case of clamped ends. Instead, we follow the approach in [7] to estimate $\frac{d}{dt}\|\nabla_t^m\vec{T}\|_{L^2}$ instead of $\frac{d}{dt}\|\nabla_s^m\vec{\kappa}\|_{L^2}$ as done in [8], where ∇_s and ∇_t are covariant differentiations defined in (6). This approach simplifies calculation of boundary terms, coming from integration by parts in the estimates of $\frac{d}{dt}\|\nabla_s^m\vec{\kappa}\|_{L^2}$. But it requires more notation in marking the higher-order derivatives of $\vec{\lambda}(t)$.

Below are the main results of this article.

Theorem 1.1. *Suppose $f_0 : \bar{I} \to \mathbb{R}^2$ is a $C^2(\bar{I})$ plane curve, parametrized by its arclength and $L > |f_0(L) - f_0(0)|$. Then, subject to the clamped boundary conditions, i.e., fixed unit tangent vectors $(f_0')_{\lfloor \partial I}$ and fixed end points $(f_0)_{\lfloor \partial I}$, there exists a long-time inextensible evolution of plane curves $f : \bar{I} \times [0,\infty) \to \mathbb{R}^2$ such that*

$$f, \partial_s f, \partial_s^2 f \in C^0(\bar{I} \times [0,\infty)) \cap C^\infty(\bar{I} \times (0,\infty)),$$

and the elastic (bending) energy decreases continuously in time, i.e., $\mathscr{E}[f](t) \searrow$ as $t \nearrow \infty$, $\forall t \in [0,\infty)$. Moreover, the inextensible evolution will sub-converge to an inextensible elastica f_∞, i.e., the curve f_∞ is an equilibrium configuration of the bending energy $\mathscr{E}$ given in (7) with the clamped boundary conditions among the class of inextensible curves $\mathscr{D}_{f_0}$ defined in (8).

The evolution of curves in Theorem 1.1 is based on the solutions of parabolic equation for the unit tangent of plane curves studied in Theorem 1.2 below.

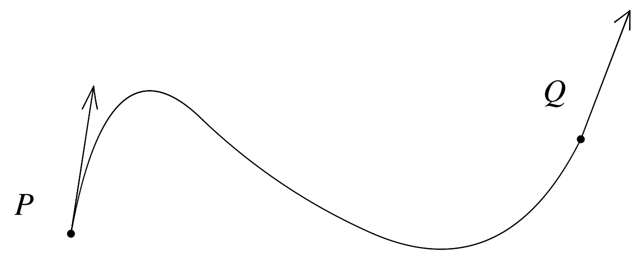

Figure 2: An initial plane curve connecting points P and Q fulfilling prescribed tangent vectors at the end points

Theorem 1.2. *Let the initial data $\vec{T}_0 : \bar{I} \to \mathbb{S}^1(1) \subset \mathbb{R}^2$ belong to the class $C^1(\bar{I})$. Then, subject to the initial-boundary conditions (2) and (3), there exists a global solution $\vec{T} : \bar{I} \times [0,\infty) \to \mathbb{S}^1(1)$ of the L^2-flow (1) such that*

$$\vec{T}, \partial_s \vec{T} \in C^0(\bar{I} \times [0,\infty)) \cap C^\infty(\bar{I} \times (0,\infty)),$$

and the elastic (bending) energy decreases continuously in time, i.e., $\mathscr{F}_{L,\Delta p}[T](t) \searrow$ as $t \nearrow \infty$, $\forall t \in [0,\infty)$. Moreover, there exists a convergent (sub-)sequence $\vec{T}_i(\cdot) := \vec{T}(\cdot, t_i)$, $t_i \in (0,+\infty)$, $t_i \to \infty$, such that $\vec{T}_\infty = \lim_{i \to \infty} \vec{T}_i$ is a critical point of the functional $\mathscr{F}_{L,\Delta p}$, defined in (10) with respect to the admissible class $\mathscr{A}_{L,0}$ in (9).

1.2 the fourth-order elastic flow

In this subsection we illustrate the other approach on elastic flow of curve/networks. This is also part of joint work with my collaborators, Prof. Anna Dall'Acqua and Prof. Paola Pozzi.

We consider the short-time and long-time evolution of an *elastic* three networks in $\mathbb{R}^n$ ($n \geq 2$) as depicted in Figure 1, that is with three fixed boundary points P_1, P_2, P_3 and one moving triple junction. That is, a three-pointed curved star with a star-center that may move in time.

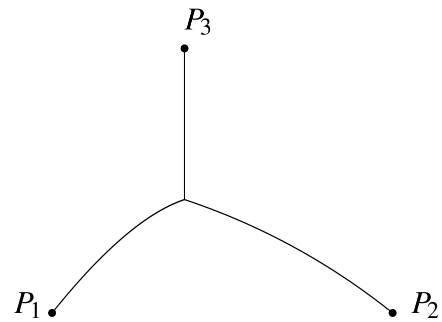

Figure 3: The model case of triple junctions

We study and give a long time existence result for elastic motion with a penalization term on the length and some extra topological conditions that prevents the appearance of some pathological cases: for instance, the triple junction should not be allowed to collapse on one of the boundary points P_i.

Before stating our main result, we introduce briefly the set up of our work and recall some well known facts.

The elastic energy of a smooth regular curve (an immersion) $f : I \to \mathbb{R}^n$, $n \geq 2$, $I = (0,1)$ is given by

$$\mathscr{E}(f) = \frac{1}{2}\int_I |\vec{\kappa}|^2 ds, \tag{16}$$

where $ds = |\partial_x f|dx$ is the arc-length element and $\vec{\kappa}$ is the curvature vector of the curve. Defining $\partial_s = |\partial_x f|^{-1}\partial_x f$, then $\vec{\kappa} = \partial_s^2 f$. The length is given by

$$\mathscr{L}(f) = \int_I ds.$$

For $\lambda \geq 0$ let

$$\mathscr{E}_\lambda(f) = \mathscr{E}(f) + \lambda \mathscr{L}(f)\,. \tag{17}$$

This is the energy that we consider: the length of the curve is allowed to change in time but its growth is penalized according to the weight λ.

Now consider three smooth regular curves $f_i : I \to \mathbb{R}^n$, $I = [0,1]$, $i = 1,2,3$, such that

1. The end-points are fixed:

$$f_1(1) = P_1,\ f_2(1) = P_2,\ f_3(1) = P_3, \tag{18}$$

with given distinct points P_i, $i = 1,2,3$, in $\mathbb{R}^n$ (recall Figure 1.2). Of course, there is a plane that contains these three points.

2. The curves start at the same point

$$f_1(0) = f_2(0) = f_3(0)\,.$$

In the following we call $\Gamma = \{f_1, f_2, f_3\}$ a *three-pointed star network* or simply *network*.

For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$, the energy of the network $\Gamma = \{f_1, f_2, f_3\}$ is given by

$$\mathscr{E}_\lambda(\Gamma) = \sum_{i=1}^{3} \mathscr{E}_{\lambda_i}(f_i)\,. \tag{19}$$

Here and in the following we agree that $\mathscr{E}(\Gamma) := \mathscr{E}_0(\Gamma)$.

We let the network Γ evolve in time according to an L^2-gradient flow for the energy $\mathscr{E}_\lambda$. Natural boundary conditions are imposed on the three curves. We are writing up the short-time existence result. Here we only list our result on the long-time existence.

Theorem 1.3. *Let $\Gamma_0 = \{f_{1,0}, f_{2,0}, f_{3,0}\}$ be a network of regular smooth curves in $\mathbb{R}^n$ such that:*

$$\begin{aligned} &f_{1,0}(1) = P_1,\ f_{2,0}(1) = P_2,\ f_{3,0}(1) = P_3,\\ &f_{1,0}(0) = f_{2,0}(0) = f_{3,0}(0),\\ &\vec{\kappa}_{i,0}(x) = 0 \textit{ for } x = 0,1 \textit{ and } i = 1,2,3 \end{aligned} \tag{20}$$

as well as

$$\sum_{i=1}^{3} (\nabla_s \vec{\kappa}_{i,0} - \lambda_i \partial_s f_{i,0})\Big|_{x=0} = 0\,. \tag{21}$$

Moreover let Γ_0 satisfy appropriate compatibility conditions, and be such that at the triple junction at least two curves form a strictly positive angle. Then the following holds:

(i) Long-time existence result: *the equations*

$$\partial_t f_i - \langle \partial_t f_i, \partial_s f_i \rangle \partial_s f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i \textit{ on } (0,T) \times I \textit{ for } i = 1,2,3, \tag{22}$$

with boundary conditions

$$\begin{cases} f_i(t,1) = P_i, & \textit{for all } t \in (0,T), i = 1,2,3,\\ \vec{\kappa}_i(t,1) = 0 = \vec{\kappa}_i(t,0) & \textit{for all } t \in (0,T),\ i = 1,2,3,\\ f_1(t,0) = f_2(t,0) = f_3(t,0) & \textit{for all } t \in (0,T),\\ \quad \textit{and } \displaystyle\sum_{i=1}^{3} (\nabla_s \vec{\kappa}_i(t,0) - \lambda_i \partial_s f_i(t,0)) = 0 & \textit{for all } t \in (0,T), \end{cases} \tag{23}$$

and initial value

$$\Gamma(t=0) = \{f_1(0,\cdot), f_2(0,\cdot), f_3(0,\cdot)\} = \Gamma_0, \textit{ in } I$$

admit a smooth global solution in time, provided that, along the flow, the lengths $\mathscr{L}(f_i)$ of the three curves are uniformly bounded from below and that the dimension of the space spanned by the unit tangents $\partial_s f_i$, $i = 1,2,3$, at the triple junction is bigger or equal to two.

(ii) Sub-convergence result: *under the mentioned hypothesis, it is possible to find a sequence of time $t_i \to \infty$ such that the networks $\Gamma(t_i)$ sub-converge, after an appropriate reparametrization, to a critical point for the energy $\mathscr{E}_\lambda(\Gamma)$ and subject to the boundary conditions given in* (20) *and* (21)*.*

The compatibility conditions follow the definition given by Solonnikov. Smooth solution means that the three parametrization of the three curves are smooth functions in the time and space variable. The extension of the long-time existence result to the case $\lambda_i \geq 0$ is also discussed.

Note that the above theorem must be understood in a *geometrical sense*: that is the existence of a global parametrization of the flow is meant up to reparametrization. So our result states that we are able to find a global in time *smooth motion* of the network, provided two topological constraints are fulfilled during the flow: namely that the lengths of the curves are uniformly bounded from below and that the the curves never entirely "collapse" to a configuration where all tangents vectors are parallel at the triple junctions. The necessity of the topological constraints occurs naturally as follows: the bound from below on the lengths of the curves is needed to apply interpolation inequalities; that the dimension of the space spanned by the unit tangents at the triple junction should always be bigger or equal to two arise when we express the tangential components at the boundary in terms of geometric quantities. At the moment we have no means to control these topological constraints: whether and how this could be done is subject to future studies.

To achieve our goal, we will consider in place of (22) equations of type

$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i + \varphi_i \partial_s f_i, \text{ on } (0,T) \times I \text{ for } i = 1,2,3,$$

where φ_i are smooth functions. Note that the presence of the tangential components is necessary in order for the flow to fulfill the topological constraint that the curves stay "glued" at the triple junction (concurrency condition), with the latter being able to move freely in time. A proper choice of tangential component is necessary and is discussed in details later.

Our strategy can be summarized as follows: starting from a short-time existence result we reparametrize the flow in such a way that for each curve the maps φ_i linearly interpolate their values between the boundary points. For this choice of parametrization we consider the long-time behavior of the network, and show that if the flow does not exist globally then we obtain a contradiction. This is achieved by obtaining uniform bounds for the curvature and its derivatives, together with a control on the arc-length, up to the maximal time of existence $0 < T < \infty$. With these estimates we are able to extend the flow smoothly up to T and then restart the flow, contradicting the maximality of T.

In its essence our proof strategy is not different from our previous works on long-time existence for open elastic curves in $\mathbb{R}^n$ ([2], [3], [4]): we use inequalities of Gagliardo-Nirenberg type, exploit the boundary conditions to reduce the order of some boundary terms, and rely heavily on interpolation estimates presented in [3]. However, the treatment of the tangential components is completely new and far from trivial. In particular the "algebra" for the maps φ_i (that is how their derivatives in time and space behave with respect to the order of the studied PDEs) must be thoroughly understood. Furthermore, an accurate choice of the "right" vector field for which uniform bounds are derived is absolutely crucial for any of the presented arguments to work. Finally, because of the interaction of the three curves proofs become increasingly technical and lengthy, and several new lemmas are derived in order to make our arguments more concise and more transparent.

References

[1] Angenent, S. On the formation of singularities in the curve shortening flow. *J. Differential Geom. 33*, 3 (1991), 601–633.

[2] Dall'Acqua, A., Lin, C.-C., and Pozzi, P. Evolution of open elastic curves in $\mathbb{R}^n$ subject to fixed length and natural boundary conditions. *Analysis (Berlin) 34*, 2 (2014), 209–222.

[3] Dall'Acqua, A., Lin, C.-C., and Pozzi, P. A gradient flow for open elastic curves with fixed length and clamped ends. *Ann.Sc.Norm.Super.Pisa Cl 17*, 5 (2017), 1031–1066.

[4] Dall'Acqua, A., and Pozzi, P. A Willmore-Helfrich L^2-flow of curves with natural boundary conditions. *Comm. Anal. Geom. 22*, 4 (2014), 617–669.

[5] Dziuk, G., Kuwert, E., and Schätzle, R. Evolution of elastic curves in $\mathbb{R}^n$: existence and computation. *SIAM J. Math. Anal. 33*, 5 (2002), 1228–1245 (electronic).

[6] Koiso, N. On the motion of a curve towards elastica. In *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, vol. 1 of *Sémin. Congr.* Soc. Math. France, Paris, 1996, pp. 403–436.

[7] Lin, C.-C. L^2-flow of elastic curves with clamped boundary conditions. *J. Differential Equations 252*, 12 (2012), 6414–6428.

[8] Lin, C.-C., Lue, Y.-K., and Schwetlick, H. R. The second-order L^2-flow of inextensible elastic curves with hinged ends in the plane. *J. Elast 119*, 2 (2015), 263–291.

[9] Novaga, M., and Okabe, S. Curve shortening-straightening flow for non-closed planar curves with infinite length. *J. Differential Equations 256*, 3 (2014), 1093–1132.

[10] Polden, A. Curves and surfaces of least total curvature and fourth-order flows. *PhD Thesis, Universität Tübingen* (1996).

[11] Wen, Y. l^2 flow of curve straightening in the plane. *Duke Math. J. 70*, 3 (1993), 683–698.

[12] Wen, Y. Curve straightening flow deforms closed plane curves with nonzero rotation number to circles. *J. Differential Equations 120*, 1 (1995), 89–107.

Report of my Academic Visiting

Supported by MoST

Visiting Period: 07.29.2019 - 08.27.2019

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1 Activities and Achievement

In 2019 summer, I visit Prof. Dr. Anna Dall'Acqua at University of Ulm for 4 weeks. During the period of time, the other collaborator of mine, Prof. Dr. Paola Pozzi also visit Anna from University of Duisburg-Essen so that we could all work/discussed together on our joint project.

The main achievement in research are listed below.

- the second-order inextensible elastic flow with clamped ends and dynamical boundary conditions in $\mathbb{R}^2$.
- the fourth-order elastic flow for networks in $\mathbb{R}^n$.

The second-order elastic flow for inextensible planar curves was first investigated by Wen (Duke Math. J. 1993). However, the work only focus on closed curves. I initiated the cases of open curves with finite total length a few years ago (with Dr. Yang-Kai Lue and Hartmut Schwetlick), in which we started from the simpler case, the so-called hinged boundary condition. The case of clamped ends is a more difficult one. Without understanding how to deal with these two boundary conditions, one can't figure out how to tackle the others, e.g., mixed-up boundary conditions and dynamical ones. What I have achieved during the period is to simplify the argument in the preprint. I reduced many arguments using potential theory in the regularity of PDE solutions to the least amount so tat the article could be kept in a shorter length. Below is the main result.

Theorem 1. *Suppose $f_0 : \bar{I} \to \mathbb{R}^2$ is a $C^2(\bar{I})$ planar curve, parametrized by its arclength and $L > |f_0(L) - f_0(0)|$. Then, subject to the clamped boundary conditions, i.e., fixed unit tangent vectors $(f_0')_{\lfloor \partial I}$ and fixed end points $(f_0)_{\lfloor \partial I}$, there exists a long-time inextensible evolution of planar curves $f : \bar{I} \times [0, \infty) \to \mathbb{R}^2$ such that*

$$f, \partial_s f, \partial_s^2 f \in C^0(\bar{I} \times [0, \infty)) \cap C^\infty(\bar{I} \times (0, \infty)),$$

and the elastic (bending) energy decreases continuously in time, i.e., $\mathcal{E}[f](t) \searrow$ as $t \nearrow \infty$, $\forall\, t \in [0, \infty)$. Moreover, the inextensible evolution will sub-converge to an inextensible elastica f_∞, i.e., the curve f_∞ is an equilibrium configuration of the bending energy $\mathcal{E}$ with the clamped boundary conditions among the class of inextensible curves $\mathcal{D}_{f_0}$ defined later.

In the second topic, which is the main joint project with Anna and Paola, focuses on the L^2-gradient flow of the penalized elastic energy on networks of q-curves in $\mathbb{R}^n$ for $q \geq 3$. Each curve is fixed at one end-point and at the other is joint to the other curves at a movable q-junction. For this geometric evolution problem with natural boundary condition we show the existence of smooth solutions for a (possibly) short interval of time. Since the geometric problem is not well-posed, due to the freedom in reparametrization of curves, we consider a fourth-order non-degenerate parabolic quasilinear system, called the analytic problem, and show first a short-time existence result for this parabolic system. The proof relies on applying Solonnikov's theory on linear parabolic systems and Banach fixed point theorem in proper Hölder spaces. Then the original geometric problem is solved by establishing the relation between the analytical solutions and the solutions to the geometrical problem. Below is one of the main results.

Theorem 2. *[Geometric existence Theorem] Let $n \geq 2$, $q \geq 3$, $\alpha \in (0,1)$ and P_i, $i \in \{1,\dots,q\}$, be given points in ^{n}. Given $f_{0,i} : [0,1] \to \mathbb{R}^n$, $f_{0,i} \in C^{4,\alpha}([0,1])$, $i \in \{1,\dots,q\}$, regular curves, satisfying the non-collinearity condition (NC), (1.1) below,*

$$\begin{cases} f_{0,i}(1) = P_i, & \text{for all } i \in \{1,\dots,q\}, \\ \vec{\kappa}_{0,i}(1) = 0 = \vec{\kappa}_{0,i}(0), & \text{for all } i \in \{1,\dots,q\}, \\ f_{0,i}(0) = f_{0,j}(0) & \text{for all } i,j \in \{1,\dots,q\}, \\ \quad \text{and } \displaystyle\sum_{i=1}^{q} (\nabla_s \vec{\kappa}_{0,i}(0) - \lambda_i \partial_s f_{0,i}(0)) = 0\,, \end{cases} \tag{1.1}$$

and

$$\nabla_s^2 \vec{\kappa}_{0,i} = 0 \qquad \text{at } x = 1, \quad i \in \{1,\dots,q\}, \tag{1.2}$$

$$-\nabla_s^2 \vec{\kappa}_{0,i} + \varphi_{0,i}\partial_s f_{0,i} = -\nabla_s^2 \vec{\kappa}_{0,j} + \varphi_{0,j}\partial_s f_{0,j} \quad \text{at } x = 0 \quad \text{for } i,j \in \{1,\dots,q\}, \tag{1.3}$$

with $\varphi_{0,i}$ defined in (1.4) below,

$$\varphi_i(t, x = 0) = \varphi_i\big(\partial_s f_j(t,0), \quad \nabla_s^2 \vec{\kappa}_j(t,0), j = 1,2,\dots,q\big) \tag{1.4}$$

then there exist $T > 0$ and regular curves $f_i \in C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times I; \mathbb{R}^n)$, $i \in \{1,\dots,q\}$, such that

$$(\partial_t f_i)^\perp = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i, \quad i \in \{1,\dots,q\},$$

together with the boundary conditions (1.5),

$$\begin{cases} f_i(t,1) = P_i, & \text{for all } t \in [0,T], i \in \{1,\dots,q\}, \\ \vec{\kappa}_i(t,1) = 0 = \vec{\kappa}_i(t,0) & \text{for all } t \in [0,T],\ i \in \{1,\dots,q\}, \\ f_i(t,0) = f_j(t,0) & \text{for all } t \in [0,T],\ i,j \in \{1,\dots,q\}, \\ \quad \text{and } \displaystyle\sum_{i=1}^{q} (\nabla_s \vec{\kappa}_i(t,0) - \lambda_i \partial_s f_i(t,0)) = 0 & \text{for all } t \in [0,T]. \end{cases} \tag{1.5}$$

and the initial condition $\Gamma = \{f_1,\dots,f_q\}|_{t=0}$ equal to $\Gamma_0 = \{f_{0,1},\dots,f_{0,q}\}$ that is

$$f_i(t=0) = f_{0,i} \circ \phi_i, \quad i \in \{1,\dots,q\}, \tag{1.6}$$

with $\phi_i \in C^{4,\alpha}([0,1],[0,1])$, orientation preserving diffeomorphisms. Moreover, we have instant parabolic smoothing, that is $f_i \in C^\infty((0,T] \times [0,1])$ for any $i \in \{1,\dots,q\}$, and the non-collinearity condition holds at the triple junction for any time $t \in [0,T]$.

107年度專題研究計畫成果彙整表

計畫主持人：林俊吉			計畫編號：107-2115-M-003-002-		
計畫名稱：座落在平面凸區域上的最小網絡之組態					
成果項目			量化	單位	質化（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文	0	篇	
		研討會論文	0		
		專書	0	本	
		專書論文	0	章	
		技術報告	0	篇	
		其他	0	篇	
國外	學術性論文	期刊論文	0	篇	
		研討會論文	0		
		專書	0	本	
		專書論文	0	章	
		技術報告	2	篇	1. title: 2nd-order elastic flow of curves with clamped ends (40 pages；submitting) 2. title: 2nd-order elastic flow of curve networks (30 pages；preprint)
		其他	0	篇	
參與計畫人力	本國籍	大專生	0	人次	
		碩士生	0		
		博士生	0		
		博士級研究人員	0		
		專任人員	0		
	非本國籍	大專生	0		
		碩士生	0		
		博士生	1		陳世勇（第4年）
		博士級研究人員	0		
		專任人員	0		
其他成果（無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）			本研究有促成國際合作、交流之功能，例如 Prof. Mateo Novaga (University of Pisa) 與 Prof. Paola Pozzi (University of Duisburg-Essen) 近期就受到我們這個研究的引導，開始研究所謂的 p-elastic flow (discrete flow)。		