科技部補助專題研究計畫報告

黎卡提微分方程之混沌動態及矩陣指數上的應用

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本研究具有政策應用參考價值：■否 □是，建議提供機關
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本研究具影響公共利益之重大發現：□否 □是

中 華 民 國 109 年 09 月 26 日

中 文 摘 要 ： 在科學計算領域，正交迭代在計算對應於最大k個特徵值的不變子空間中起著至關重要的作用。 在本文中，我們構造了一個流，該流連接由正交迭代生成的矩陣序列。 這樣的流稱為正交流。 此外，我們還表明，正交迭代形成了正交流的時間一映射。 通過使用適當的變量更改，可以將正交流轉換為Riccati微分方程（RDE）。 相反，RDE也可以轉換為可以用正交流乘以正交矩陣表示的流。

中文關鍵詞： 正交迭代，不變子空間，正交流，黎卡提微分方程

英 文 摘 要 ： In the fi eld of scienti c computation, the orthogonal iteration plays an essential role in computing the invariant subspace corresponding to the largest k eigenvalues. In this paper, we construct a flow that connects the sequence of matrices generated by the orthogonal iteration. Such a flow is called an orthogonal ow. Besides, we also show that the orthogonal iteration forms a time-one mapping of the orthogonal flow. By using a suitable change of variables, the orthogonal flow can be transformed into a Riccati differential equation (RDE). Conversely, an RDE also can be transformed into a flow that can be represented by the orthogonal flow multiplied by an orthogonal matrix.

英文關鍵詞： orthogonal iteration, invariant subspace, orthogonal flow, Riccati differential equation (RDE)

The orthogonal flows for orthogonal iteration

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Abstract

In the field of scientific computation, the orthogonal iteration plays an essential role in computing the invariant subspace corresponding to the largest k eigenvalues. In this paper, we construct a flow that connects the sequence of matrices generated by the orthogonal iteration. Such a flow is called an orthogonal flow. Besides, we also show that the orthogonal iteration forms a time-one mapping of the orthogonal flow. By using a suitable change of variables, the orthogonal flow can be transformed into a Riccati differential equation (RDE). Conversely, an RDE also can be transformed into a flow that can be represented by the orthogonal flow multiplied by an orthogonal matrix.

AMS classification: 15A23, 15A24, 34A12, 37N30, 47J25, 65F30.

Keywords: orthogonal iteration, invariant subspace, orthogonal flow, Riccati differential equation (RDE).

1 Introduction

The investigation for subspace iterations plays a vital role in the field of matrix computation [9]. Let A be an $n \times n$ nonsingular matrix, and Y_0 be an $n \times r$ matrix ($n \geq r$) with r orthonormal columns. The orthogonal iteration starts with an orthonormal matrix Y_0. It generates a sequence $\{Y_k\}$ by the recurrence formula

$$Y_{k+1}R_{k+1} = AY_k. \tag{1.1}$$

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Here, the matrix R_{k+1} is upper triangular with positive diagonal entries, and the matrix Y_{k+1} has orthonormal columns. Namely, $Y_{k+1}R_{k+1}$ forms a QR factorization of AY_k. The space spanned by Y_k will asymptotically converge to the r-dimensional invariant subspace of A associated with the r eigenvalues of A with the largest modulus. The orthogonal iteration is the generalization of the power method. Finding a smooth curve with a specific structure that passes through a sequence of iteration generated by some numerical algorithm is a popular topic studied by many researchers. It is mainly in the study of the so-called Toda flow that links matrices/matrix pairs generated by the QR/QZ algorithm [3, 4, 17]. In [5, 6], the authors study a general framework for constructing isospectral flows in the space of $n \times n$ matrices and characterize their asymptotic behavior. The isospectral flows can generate sequences of iterative processes associated with abstract matrix factorizations. This general framework can be practically applied to unify three well-known matrix factorization techniques which are used in numerical linear algebra and suggests some new matrix factorization. In [11], a structure-preserving flow is investigated, which passes through a sequence of iterates generated by the structure-preserving doubling algorithms. The structure-preserving doubling algorithms are employed for solving the stabilizing solutions of Riccati-type equations. The asymptotic behavior of the structure-preserving flow is studied in [12]. Based on the flow, authors of [13] developed a numerical method that took care of the symplectic structure for computing Hamiltonian matrix exponential. Motivated by those previous works, we are interested in the construction and analysis of the flows related to the orthogonal iteration in this paper.

Our main contribution is to construct a nonlinear differential equation associated with the flow passing through the orthogonal iterations. Specifically, let $A \in \mathbb{R}^{n\times n}$ be nonsingular, and $B \in \mathbb{R}^{n\times n}$ be the matrix such that $e^B = A$. Consider the initial-value problem (IVP)

$$\dot{Y} = BY + Y(-\frac{1}{2}Y^\top(B^\top + B)Y + K_Y), \quad Y(0) = Y_0, \tag{1.2a}$$

where the initial matrix $Y_0 \in \mathbb{R}^{n\times r}$ with $n \geqslant r$ is orthogonal, that is, $Y_0^\top Y_0 = I_r$, and the matrix $K_Y \in \mathbb{R}^{r\times r}$ is defined as

$$(K_Y)_{ij} = \begin{cases} \frac{1}{2}(Y^\top(B^\top + B)Y)_{ij} & \text{if } i > j, \\ 0 & \text{if } i = j, \\ -\frac{1}{2}(Y^\top(B^\top + B)Y)_{ij} & \text{if } i < j. \end{cases} \tag{1.2b}$$

Here, K_Y is skew-symmetric. We show that the solution $Y(t)$ of (1.2) exists and is unique for all time $t \in \mathbb{R}$ and it preserves the orthogonality, i.e., $Y(t)^\top Y(t) = I$. In this paper, the unique solution $Y(t)$ of (1.2) is called the *orthogonal flow*. When sampled at positive integer times, the orthogonal flow gives the same sequence of matrices generated by the orthogonal iteration with $A = e^B$. In other words, the orthogonal iteration forms a time-one mapping of the orthogonal flow.

The Riccati differential equation (RDE) is the quadratic differential equation that plays a role in a wide variety of applications for science and applied mathematics, for example in optimal control theory [2, 14, 15, 16] or two-point boundary value problems [7, 8]. There is a close relationship between the orthogonal flow (1.2) and an RDE. Furthermore, our second contribution shows that the orthogonal flow (1.2) can be transformed into a solution of an RDE. Conversely, a solution of an RDE also can be transformed into a flow of the form $Y(t)Q(t)$ in which $Q(t)$ is a specific orthogonal matrix. As a result, the orthogonal flow can be used to define the extended solution of an RDE and be further applied to compute the extended solution of an RDE numerically.

The organization of this paper is as follows. In section 2, some preliminaries are given. In section 3, we study the fundamental properties of the orthogonal flow (1.2), such as the existence and uniqueness of the solution to (1.2) and the orthonormalization of its solution. Besides, we also show that the flow connects the sequence of matrices generated by the orthogonal iteration (1.1) in which $A = e^B$ and Y_0 are provided. In section 4, a generalized orthogonal flow is defined. In section 5, we study the relationship between the orthogonal flow and the RDE. Some concluding remarks are given in section 6.

2 Preliminaries

In this section, we introduce notations, definitions and some preliminary results. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of A. The matrix A is called orthogonal if $A^\top A = I_m$, i.e., the columns of A are orthonormal. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we use the notation $A \succ 0$ (or $A \succeq 0$) to denote that A is positive definite (or positive semi-definite). A matrix function $A(t)$ for $t \in (a, b)$ is said to be of class $\mathcal{C}^1$ if each entry function of $A(t)$ is continuously differentiable for $t \in (a, b)$. We use capital letters to denote matrices and lowercase (bold) letters to denote scalars (vectors).

Proposition 2.1. *Let $C \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. Then there exists a unique matrix $R \in \mathbb{R}^{n\times n}$ that satisfies*

(i) R is upper triangular;

(ii) the diagonals of R are all positive, i.e., $(R)_{ii} > 0$, $1 \le i \le n$,

such that

$$RR^\top = C.$$

In addition, if $C = C(t)$ is of class $\mathcal{C}^1$ on (a, b), then $R = R(t)$ is also of class $\mathcal{C}^1$ on (a, b).

Proof. Let $P = [\mathbf{e}_n, \mathbf{e}_{n-1}, \cdots, \mathbf{e}_1]$ be a permutation matrix, where $\mathbf{e}_i$ is the ith column vector of the identity matrix I_n. The matrix C is symmetric and positive definite, so is $P^\top CP$. Then the positive definite matrix $P^\top CP$ has a unique Cholesky factorization [10, p.407], i.e., $P^\top CP = LL^\top$, where L is a lower triangular matrix whose diagonal entries are real and positive. Denote $R = PLP^\top$. Then it is easy to see that R is an upper triangular matrix with $R_{i,i} = L_{n+1-i,n+1-i} > 0$ and $C = PLL^\top P^\top = PLP^\top PL^\top P^\top = RR^\top$. Since P is invertible, the upper triangular matrix R is unique.

Since $C(t) \succ 0$ is of class $\mathcal{C}^1$, it follows that $\widehat{C}(t) \triangleq P^\top C(t)P \succ 0$ is also of class $\mathcal{C}^1$. Let $R(t)$ be the unique upper triangular matrix with a positive diagonal such that $C(t) = R(t)R(t)^\top \in \mathbb{R}^{n\times n}$. Then we have $R(t) = PL(t)P^\top$ in which $L(t)$ with a positive diagonal is unique by Cholesky factorization of $\widehat{C}(t)$. Hence, it suffices to show that $L(t)$ is of class $\mathcal{C}^1$ on (a, b). The proof will be carried out by induction on n. For $n = 1$, $\widehat{C}(t) > 0$, $t \in (a, b)$, is a real-valued function of $\mathcal{C}^1$. Then $L(t) = \sqrt{\widehat{C}(t)}$ is also of class $\mathcal{C}^1$ on (a, b). This gives the desired result.

Now suppose that the statement is true for some integer $n \geqslant 1$. We prove that the statement is also true for $n+1$. Let $\widehat{C}(t) \in \mathbb{R}^{(n+1)\times(n+1)}$ be symmetric, positive definite and of $\mathcal{C}^1$ on (a, b). Denote Cholesky factorization of $\widehat{C}(t)$ by

$$\widehat{C}(t) = L(t)L(t)^\top,\ t \in (a, b), \tag{2.1}$$

where $L(t)$ is lower triangular and has positive diagonal entries. We partition $\widehat{C}(t)$ and $L(t)$ as

$$\widehat{C}(t) = \begin{bmatrix} \widehat{c}_{11}(t) & \widehat{\mathbf{c}}_{12}^\top(t) \\ \widehat{\mathbf{c}}_{12}(t) & \widehat{C}_{22}(t) \end{bmatrix} \text{ and } L(t) = \begin{bmatrix} l_{11}(t) & \mathbf{0}^\top \\ \mathbf{l}_{12}(t) & L_{22}(t) \end{bmatrix}, \tag{2.2}$$

where $\widehat{c}_{11}(t), l_{11}(t) \in \mathbb{R}$, $\widehat{\mathbf{c}}_{12}(t), \mathbf{l}_{12}(t) \in \mathbb{R}^n$ and $\widehat{C}_{22}(t), L_{22}(t) \in \mathbb{R}^{n\times n}$. From (2.1) and (2.2), we have

$$\begin{bmatrix} \widehat{c}_{11}(t) & \widehat{\mathbf{c}}_{12}^\top(t) \\ \widehat{\mathbf{c}}_{12}(t) & \widehat{C}_{22}(t) \end{bmatrix} = \begin{bmatrix} l_{11}^2(t) & l_{11}\mathbf{l}_{12}(t)^\top \\ l_{11}\mathbf{l}_{12}(t) & L_{22}(t)L_{22}(t)^\top \end{bmatrix}, \tag{2.3}$$

which implies

$$\begin{aligned} l_{11}(t) &= \sqrt{\widehat{c}_{11}(t)}\ (> 0), \\ \mathbf{l}_{12}(t) &= \frac{1}{l_{11}(t)}\widehat{\mathbf{c}}_{12}(t), \\ \widehat{C}_{22}(t) &= L_{22}(t)L_{22}(t)^\top. \end{aligned}$$

Due to $\widehat{C}(t) \in \mathcal{C}^1$, we have $l_{11}(t) \in \mathcal{C}^1$, and so $\mathbf{l}_{12}(t) \in \mathcal{C}^1$. It is also noted that $\widehat{C}_{22}(t) \in \mathbb{R}^{n\times n}$, $\widehat{C}_{22}(t) \succ 0$ and $\widehat{C}_{22}(t) = L_{22}(t)L_{22}(t)^\top$ is Cholesky factorization of $\widehat{C}_{22}(t)$. Then by the assumption of the induction, we have $L_{22}(t) \in \mathcal{C}^1$. Consequently, $L(t)$ is shown to be of class $\mathcal{C}^1$ on (a, b) and the argument for the induction is completed. □

The following proposition is straightforward, and we omit the proof.

Proposition 2.2. *Let $C \in \mathbb{R}^{n\times n}$ be symmetric. If $X \in \mathbb{R}^{n\times n}$ is a solution of the equation*

$$X + X^\top = C, \tag{2.4}$$

then the general solution of (2.4) *is*

$$X = \frac{1}{2}C + K,$$

where K is an arbitrary $n \times n$ skew-symmetric matrix.

For a matrix $A \in \mathbb{R}^{n\times m}$, the QR factorization of A is $A = QR$, where $Q \in \mathbb{R}^{n\times m}$ has orthogonal columns and $R \in \mathbb{R}^{m\times m}$ is an upper triangular matrix. In addition, if A is of full column rank and R is chosen with positive diagonal entries, then the factors Q and R are both unique.

Proposition 2.3. *Let $X \in \mathbb{R}^{n\times r}$ be of full column rank. Then there exist a unique orthogonal matrix $Y \in \mathbb{R}^{n\times r}$ and a unique upper triangular matrix $R \in \mathbb{R}^{r\times r}$ with a positive diagonal such that $X = YR^{-1}$. In addition, if $X = X(t)$ is of class $\mathcal{C}^1$ on (a, b), then $R = R(t)$ is also of class $\mathcal{C}^1$ on (a, b).*

Proof. The unique QR factorization, $X = YR^{-1}$, is well-known and the proof can be found in [10, p.112]. Suppose that $X = X(t)$ is of full column rank and is of class $\mathcal{C}^1$ on (a, b). Then $C(t) = X(t)^\top X(t) \succ 0$ is also of class $\mathcal{C}^1$ on (a, b). Since $R(t)^{-1}$ is the unique upper triangular matrix with a positive diagonal such that

$$R(t)^{-\top}R(t)^{-1} = R(t)^{-\top}Y(t)^\top Y(t)R(t)^{-1} = C(t), \text{ for } t \in (a, b),$$

it follows from Proposition 2.1 that $R(t)^{-1}$ is a $\mathcal{C}^1$ function. Hence $R = R(t)$ is also of class $\mathcal{C}^1$ on (a, b). □

3 The connection with orthogonal iteration

We first define the linear differential equation on $\mathbb{R}^{n\times r}$,

$$\dot{X} = BX \text{ with } X(0) = Y_0, \tag{3.1}$$

where $Y_0 \in \mathbb{R}^{n\times r}$ has orthonormal columns. Consequently, the solution of (3.1) is

$$X(t) = e^{Bt}Y_0. \tag{3.2}$$

Since Y_0 is of full column rank and e^{Bt} is invertible, $X(t)$ is also of full column rank for each $t \in \mathbb{R}$. Hence, $X(t)^\top X(t) \in \mathbb{R}^{r\times r}$ is positive definite and is of class $\mathcal{C}^1$. It follows from Proposition 2.1 that there is a unique upper triangular matrix $R_X(t) \in \mathbb{R}^{r\times r}$ with a positive diagonal such that

$$R_X(t)R_X(t)^\top = (X(t)^\top X(t))^{-1} \text{ for } t \in \mathbb{R}. \tag{3.3}$$

Here, $R_X(t) \in \mathbb{R}^{r\times r}$ is a $\mathcal{C}^1$ function on $\mathbb{R}$. Note that $X(0) = Y_0$ is orthogonal, i.e., $Y_0^\top Y_0 = I$. Consequently, $R_X(0) = I$. Let

$$Y(t) = X(t)R_X(t) \text{ for } t \in \mathbb{R}. \tag{3.4}$$

Then $Y(t)$ is of class $\mathcal{C}^1$ and $Y(0) = Y_0$. In this section, we shall demonstrate that $Y(t)$ is the unique solution of IVP (1.2) and demonstrate that $Y(t)$ links the sequence of matrices generated by orthogonal iteration.

The following lemma is related to the derivative of the function $R_X(t)$ and will be used in the proof of Proposition 3.1.

Lemma 3.1. *Let $R_X(t)$ and $Y(t)$ be the matrices defined in* (3.3) *and* (3.4)*, respectively. Then*

$$R_X^{-1}\dot{R}_X = -\frac{1}{2}Y^\top(B^\top + B)Y + K_Y,$$

where $K_Y = K_Y(t)$ is the skew-symmetric matrix defined in (1.2b)*.*

Proof. Taking derivatives on both sides of (3.3), it follows from (3.1) and (3.3) that

$$\begin{aligned}\dot{R}_X R_X^\top + R_X\dot{R}_X^\top &= -(X^\top X)^{-1}(\dot{X}^\top X + X^\top\dot{X})(X^\top X)^{-1}\\ &= -R_XR_X^\top(X^\top B^\top X + X^\top BX)R_XR_X^\top.\end{aligned}$$

Snice R_X is invertible, it follows from (3.4) that

$$R_X^{-1}\dot{R}_X + \dot{R}_X^\top R_X^{-\top} = -R_X^\top X^\top(B^\top + B)XR_X = -Y^\top(B^\top + B)Y.$$

Note that R_X is upper triangular and so is $R_X^{-1}\dot{R}_X$. Then Proposition 2.2 can be used to obtain

$$R_X^{-1}\dot{R}_X = -\frac{1}{2}Y^\top(B^\top + B)Y + K_Y,$$

where K_Y defined in (1.2b) is the skew-symmetric matrix which enforces the matrix $\frac{1}{2}Y^\top(B^\top + B)Y + K_Y$ to be upper triangular. □

Next, we show that $Y(t)$ in (3.4) is a solution of the IVP (1.2).

Proposition 3.1. *The matrix function $Y(t)$ in* (3.4) *is a solution of the IVP* (1.2) *with $Y(t)^\top Y(t) = I$ for all $t \in \mathbb{R}$.*

Proof. We first show that $Y(t)$ is orthogonal for all $t \in \mathbb{R}$. From (3.3), we have

$$Y(t)^\top Y(t) = R_X(t)^\top X(t)^\top X(t)R_X(t) = R_X(t)^\top(R_X(t)R_X(t)^\top)^{-1}R_X(t) = I,$$

for all $t \in \mathbb{R}$. Hence, $Y(t)$ is orthogonal.

Next, we show that $Y(t)$ satisfies the IVP (1.2). Taking the derivative of $Y(t)$, it follows from Lemma 3.1 and (3.1) that

$$\begin{aligned}\dot{Y} &= \dot{X}R_X + X\dot{R}_X = BXR_X + XR_XR_X^{-1}\dot{R}_X \\ &= BY + Y(-\frac{1}{2}Y^\top(B^\top + B)Y + K_Y),\end{aligned}$$

where K_Y is defined in (1.2b). Moreover, $Y(0) = Y_0$. Hence $Y(t)$ is a solution of the IVP (1.2). □

In the following theorem, we prove our first main result in which the existence and uniqueness of the solution $Y(t)$ for IVP (1.2) can be obtained. We also show that $Y(t)$ links the sequence generated by the orthogonal iteration.

Theorem 3.2. *The orthogonal flow $Y(t)$ in* (3.4) *is the unique solution of IVP* (1.2). *In addition, for each integer k, $Y(k) = Y_k$, where $Y_k \in \mathbb{R}^{n\times r}$ is the matrix generated by the orthogonal iteration with Y_0 as its input and $A = e^B$.*

Proof. The existence and orthogonality of the solution $Y(t)$ for the IVP (1.2) can be obtained by Proposition 3.1. Note that $\|Y(t)\|_F = \sqrt{r}$ for $t \in \mathbb{R}$, where $\|\cdot\|_F$ is the Frobenius norm. Because $F(Y) = BY + Y(-\frac{1}{2}Y^\top(B^\top + B)Y + K_Y)$, is a polynomial function, it is Lipschitz continuous on $\mathcal{B} = \{Y \in \mathbb{R}^{n\times r} \mid \|Y\|_F \leqslant 2\sqrt{r}\}$ which contains the solution $Y(t)$ in (3.4). Therefore, the solution $Y(t)$ for IVP (1.2) is unique. From (3.2) and (3.4), we have

$$Y(1) = X(1)R_X(1) = e^BY_0R_X(1) = AY_0R_X(1). \tag{3.5}$$

Since A is nonsingular and Y_0 is of full column rank, AY_0 is of full column rank. Then by Proposition 2.3 and (3.5), $Y(1)$ is the unique orthogonal factor of matrix AY_0. Therefore, $Y(1) = Y_1$, where Y_1 is the first orthogonal matrix computed by orthogonal iteration. Since the ODE (1.2) is autonomous, an inductive argument leads to $Y(k) = Y_k$ for each $k \in \mathbb{N}$. $\square$

4 The generalized orthogonal flow

In this section, we consider the general form of the matrix flow $\widehat{Y}(t)$ for $t \in \mathbb{R}$ which has orthogonal columns and has the same column space of $X(t) = e^{Bt}Y_0 \in \mathbb{R}^{n\times r}$. This flow $\widehat{Y}(t)$ is called the generalized orthogonal flow. Since $X(t)$ is of full column rank, $X(t)^\top X(t) \in \mathbb{R}^{r\times r}$ is positive definite. Consider the factorization of the positive definite matrix $(X(t)^\top X(t))^{-1}$

$$\Pi_X(t)\Pi_X(t)^\top = (X(t)^\top X(t))^{-1}, \tag{4.1}$$

where $\Pi_X(t) \in \mathbb{R}^{r\times r}$ is invertible. Recall that the upper triangular matrix $R_X(t)$ given in (3.3) with a positive diagonal is a solution of the factorization equation (4.1). The following lemma characterizes the general solution of (4.1).

Lemma 4.1. *Let $R_X(t) \in \mathbb{R}^{r\times r}$ with a positive diagonal be the unique upper triangular solution of* (3.3)*. Then the general solution of* (4.1) *is*

$$\Pi_X(t) = R_X(t)Q(t), \tag{4.2}$$

where $Q(t) \in \mathbb{R}^{r\times r}$ is an orthogonal matrix.

Proof. Let $Q(t)$ be an orthogonal matrix and $\Pi_X(t) \triangleq R_X(t)Q(t)$. Then

$$\Pi_X(t)\Pi_X(t)^\top = R_X(t)Q(t)Q(t)^\top R_X(t)^\top = (X(t)^\top X(t))^{-1}.$$

So, $\Pi_X(t)$ is a solution of (4.1). Next, we claim that any solution $\Pi_X(t)$ of (4.1) is of the form (4.2). Due to $\Pi_X(t)\Pi_X(t)^\top = R_X(t)R_X(t)^\top = (X(t)^\top X(t))^{-1}$, we have $R_X(t)^{-1}\Pi_X(t)\Pi_X(t)^\top R_X(t)^{-\top} = I$. This implies $R_X(t)^{-1}\Pi_X(t)$ is an orthogonal matrix, say $Q(t)$. Hence $\Pi_X(t) = R_X(t)Q(t)$. $\square$

Assume that $\Pi_X(t)$ satisfying (4.1) is a $\mathcal{C}^1$ function on $\mathbb{R}$. Let

$$\widehat{Y}(t) = X(t)\Pi_X(t) \text{ for } t \in \mathbb{R}, \tag{4.3}$$

where $X(t)$ is the solution of IVP (3.1). Then $\widehat{Y}(t) \in \mathbb{R}^{n\times r}$ and $X(t)$ have the same column space. In addition, $\widehat{Y}(t)$ is orthogonal because

$$\begin{aligned}\widehat{Y}(t)^\top \widehat{Y}(t) &= \Pi_X(t)^\top (X(t)^\top X(t))\Pi_X(t)\\ &= \Pi_X(t)^\top (\Pi_X(t)\Pi_X(t)^\top)^{-1}\Pi_X(t) = I.\end{aligned}$$

Using a similar technique as the proof of Lemma 3.1, we take derivatives on both sides of equation (4.1). Then we have

$$\begin{aligned}\dot{\Pi}_X\Pi_X^\top + \Pi_X\dot{\Pi}_X^\top &= -(X^\top X)^{-1}(\dot{X}^\top X + X^\top \dot{X})(X^\top X)^{-1}\\ &= -\Pi_X\Pi_X^\top (X^\top B^\top X + X^\top BX)\Pi_X\Pi_X^\top.\end{aligned}$$

Since $\Pi_X \in \mathbb{R}^{r\times r}$ is invertible, it follows from (4.3) that

$$\Pi_X^{-1}\dot{\Pi}_X + \dot{\Pi}_X^\top \Pi_X^{-\top} = -\widehat{Y}^\top (B^\top + B)\widehat{Y}.$$

Using the assumption that $\Pi_X(t)$ is a $\mathcal{C}^1$ function on $\mathbb{R}$, it follows from Proposition 2.2 that there exists a continuous skew-symmetric matrix function $K = K(t)$ such that

$$\Pi_X^{-1}\dot{\Pi}_X = -\frac{1}{2}\widehat{Y}^\top (B^\top + B)\widehat{Y} + K. \tag{4.4}$$

Next, taking derivatives on the flow $\widehat{Y}(t)$ in (4.3) and using (4.4), we have

$$\begin{aligned}\dot{\widehat{Y}} &= \dot{X}\Pi_X + X\dot{\Pi}_X\\ &= BX\Pi_X + X\Pi_X(-\frac{1}{2}\widehat{Y}^\top (B^\top + B)\widehat{Y} + K)\\ &= B\widehat{Y} + \widehat{Y}(-\frac{1}{2}\widehat{Y}^\top (B^\top + B)\widehat{Y} + K),\end{aligned}$$

where $K = K(t)$ is a continuous and skew-symmetric matrix function. To distinguish the flow $\widehat{Y}(t)$ in (4.3) from the orthogonal flow $Y(t)$ in (3.4), i.e., the solution of the IVP (1.2), we define a generalized orthogonal flow as follows.

Definition 4.1. Let $K = K(t)$ be a continuous skew-symmetric matrix function on (a, b) containing 0 and $\widehat{Y}_0^\top \widehat{Y}_0 = I$. The solution of the IVP

$$\dot{\widehat{Y}} = B\widehat{Y} + \widehat{Y}(-\frac{1}{2}\widehat{Y}^\top (B^\top + B)\widehat{Y} + K), \quad \widehat{Y}(0) = \widehat{Y}_0 \tag{4.5}$$

is called a generalized orthogonal flow.

In the following, we shall show that the IVP (4.5) has a unique solution for a given continuous skew-symmetric matrix function $K(t)$.

Theorem 4.1. *Suppose $K(t) \in \mathbb{R}^{r\times r}$ is a continuous function with $K(t)^\top = -K(t)$ defined on an open interval (a, b) containing 0. Then the following statements hold.*

(i) The IVP (4.5) has a unique solution on (a, b).

(ii) Let $\widehat{Y}(t)$ be the solution of IVP (4.5) and $Y(t)$ be the solution of IVP (1.2). Suppose that $\widehat{Y}_0 = Y_0 Q_0$, for some orthogonal matrix $Q_0 \in \mathbb{R}^{r\times r}$. Then there is a $\mathcal{C}^1$ function $Q(t) \in \mathbb{R}^{r\times r}$ such that

$$\widehat{Y}(t) = Y(t)Q(t) \text{ for } t \in (a, b). \tag{4.6}$$

More precisely, $Q(t)$ is an orthogonal matrix satisfying the linear equation

$$\dot{Q} = -K_Y Q + QK(t), \quad Q(0) = Q_0, \tag{4.7}$$

where K_Y is defined in (1.2b).

Remark 4.1. *It is also noted from (4.6) that the generalized orthogonal flow $\widehat{Y}(t)$ and the orthogonal flow $Y(t)$ span the same subspace in $\mathbb{R}^n$.*

To prove Theorem 4.1, we need the following two propositions.

Proposition 4.2. *Suppose $K(t) \in \mathbb{R}^{r\times r}$ is a continuous function with $K(t)^\top = -K(t)$ defined on an open interval (a, b) containing 0. Then the IVP*

$$\dot{V} = -K(t)V + VK(t), \quad V(0) = I \tag{4.8}$$

has a unique solution $V(t) = I$ for all $t \in (a, b)$.

Proof. Equation (4.8) is a linear ODE with continuous coefficient $K(t)$ on (a, b). Hence there exists a unique solution for the IVP (4.8). It can be found that $V(t) = I$ is the unique solution. □

Proposition 4.3. *Suppose $K(t) \in \mathbb{R}^{r\times r}$ is a continuous function with $K(t)^\top = -K(t)$ defined on an open interval (a, b) containing 0. Let K_Y be the skew-symmetric matrix defined in (1.2b), depending on the orthogonal flow $Y(t)$. Then the IVP (4.7) has a unique solution $Q(t)$ defined for all $t \in (a, b)$. Moreover, $Q(t)^\top Q(t) = I_r$ for all $t \in (a, b)$.*

Proof. Since K_Y and $K(t)$ are both continuous functions of t on (a,b), the existence and uniqueness of the solution for the linear IVP (4.7) are guaranteed. Moreover, we have

$$\begin{aligned}\frac{d}{dt}(Q(t)^\top Q(t)) &= \dot{Q}^\top Q + Q^\top \dot{Q}\\ &= (-K_Y Q + QK)^\top Q + Q^\top(-K_Y Q + QK)\\ &= -KQ^\top Q + Q^\top QK.\end{aligned}$$

The last equality holds because $K_Y^\top = -K_Y$ and $K^\top = -K$. Set $V = Q^\top Q$. Then by Proposition 4.2, we have the orthogonality of $Q(t)$, i.e. $Q(t)^\top Q(t) = I$ for all $t \in (a,b)$. □

Proof of Theorem 4.1. From Proposition 4.3, the IVP (4.7) has unique solution $Q(t)$ which is orthogonal. Let $\widehat{Y}(t) = Y(t)Q(t)$ for $t \in (a,b)$. Then $\widehat{Y}(0) = Y_0 Q_0 = \widehat{Y}_0$. We first show that $\widehat{Y}(t)$ is a solution of IVP (4.5). This is true from the following calculation:

$$\begin{aligned}\frac{d}{dt}\widehat{Y}(t) &= \frac{d}{dt}(YQ) = \dot{Y}Q + Y\dot{Q}\\ &= BYQ + Y\left(-\frac{1}{2}Y^\top(B^\top + B)Y + K_Y\right)Q + Y\left(-K_Y Q + QK(t)\right)\\ &= B\widehat{Y} + \widehat{Y}\left(-\frac{1}{2}\widehat{Y}^\top(B^\top + B)\widehat{Y} + K(t)\right).\end{aligned}$$

This implies that $\widehat{Y}(t)$ is a solution of the IVP (4.5).

Next, we claim that the IVP (4.5) has a unique solution. Let

$$\begin{aligned}\Phi(t,\widehat{Y}) &= -\frac{1}{2}\widehat{Y}^\top(B^\top + B)\widehat{Y} + K(t),\\ F(t,\widehat{Y}) &= B\widehat{Y} + \widehat{Y}\Phi(\widehat{Y}, K(t)),\\ D &= \left\{(t,\widehat{Y}) \mid t \in J, \|\widehat{Y}\|_F \le 2\sqrt{r}\right\},\end{aligned}$$

where J is a finite closed subinterval of (a,b) and contains 0 in its interior. It is noted that D is a compact set, $(0,\widehat{Y}_0) \in D$ and $\Phi(t,\widehat{Y})$ is a continuous function. Then for $(t,\widehat{Y}_1),(t,\widehat{Y}_2) \in D$, using the fact that $\|AB\|_F \leqslant \|A\|_F\|B\|_F$ for any matrices A and B (see [10, p. 291]), we have

$$\begin{aligned}&\|F(t,\widehat{Y}_1) - F(t,\widehat{Y}_2)\|_F = \|B\widehat{Y}_1 + \widehat{Y}_1\Phi(t,\widehat{Y}_1) - B\widehat{Y}_2 - \widehat{Y}_2\Phi(t,\widehat{Y}_2)\|_F\\ &\le \|B(\widehat{Y}_1 - \widehat{Y}_2)\|_F + \|(\widehat{Y}_1 - \widehat{Y}_2)\Phi(t,\widehat{Y}_1)\|_F + \|\widehat{Y}_2\left(\Phi(t,\widehat{Y}_1) - \Phi(t,\widehat{Y}_2)\right)\|_F\\ &\le \|B\|_F\|\widehat{Y}_1 - \widehat{Y}_2\|_F + \|\Phi(t,\widehat{Y}_1)\|_F\|(\widehat{Y}_1 - \widehat{Y}_2)\|_F + 2\sqrt{r}\|\Phi(t,\widehat{Y}_1) - \Phi(t,\widehat{Y}_2)\|_F.\end{aligned}$$

In addition,

$$\|\Phi(t,\widehat{Y}_1)-\Phi(t,\widehat{Y}_2)\|_F \le \frac{1}{2}\left(\|(\widehat{Y}_1^\top-\widehat{Y}_2^\top)(B^\top+B)\widehat{Y}_1\|_F+\|\widehat{Y}_2^\top(B^\top+B)(\widehat{Y}_1-\widehat{Y}_2)\|_F\right)$$
$$\le 2\sqrt{r}\|(B^\top+B)\|_F\|\widehat{Y}_1-\widehat{Y}_2\|_F.$$

By the continuity of $\Phi(t,\widehat{Y})$ on the compact set D, $\|\Phi(t,\widehat{Y}_1)\|_F$ is bounded, and hence,

$$\|F(t,\widehat{Y}_1)-F(t,\widehat{Y}_2)\|_F \le C\|\widehat{Y}_1-\widehat{Y}_2\|_F$$

for some constant C, which means that $F(t,\widehat{Y})$ is a Lipschitz continuous function of $\widehat{Y}$ on the domain D. Hence, the IVP (4.5) has a unique solution.

Let $\widehat{Y}(t)$ for $t\in(a,b)$ be defined in (4.6). Therefore, $(t,\widehat{Y}(t))\in D$ for $t\in J$. By the uniqueness of the solution to IVP (4.5), we have $\widehat{Y}(t)=Y(t)Q(t)$ with $Q(t)^\top Q(t)=I$ for all $t\in J$. Here, J can be any finite closed subinterval of (a,b). As a result, $\widehat{Y}(t)=Y(t)Q(t)$ is the unique orthogonal solution defined for all $t\in(a,b)$. □

5 The relation with RDE

We consider the Riccati differential equation (RDE):

$$\dot{W}=[-W\mid I]B\begin{bmatrix} I \\ W\end{bmatrix}$$
$$=B_{21}-WB_{11}+B_{22}W-WB_{12}W,\quad W(0)=W_0, \tag{5.1}$$

where $W=W(t)\in\mathbb{R}^{(n-r)\times r}$ and $B=\begin{bmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$ with $B_{11}\in\mathbb{R}^{r\times r}$, $B_{12}\in\mathbb{R}^{r\times(n-r)}$, $B_{21}\in\mathbb{R}^{(n-r)\times r}$, and $B_{22}\in\mathbb{R}^{(n-r)\times(n-r)}$ being constant matrices. In this section, we shall investigate the relation of the generalized orthogonal flow with the solution of RDE.

Theorem 5.1. *Let $Y(t)=\begin{bmatrix} Y_1(t)\\ Y_2(t)\end{bmatrix}$ for $t\in\mathbb{R}$ be the solution of IVP (1.2) with $Y_1(t)\in\mathbb{R}^{r\times r}$ and $Y_2(t)\in\mathbb{R}^{(n-r)\times r}$. Suppose that $Y_1(0)$ is nonsingular. Let $(a,b)\subseteq\mathbb{R}$ be the connected component of the open set $\{t\in\mathbb{R}\mid \det Y_1(t)\neq 0\}$ that contains 0 and let $W(t)=Y_2(t)Y_1(t)^{-1}$ for $t\in(a,b)$. If $W_0=Y_2(0)Y_1(0)^{-1}$, then $W(t)$ is the solution of RDE (5.1). In addition, the interval (a,b) is the maximal interval of existence of the solution of RDE (5.1).*

Proof. Since $Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$ is the solution of the IVP (1.2) and $W(t) = Y_2(t)Y_1(t)^{-1}$, we have $[-W(t), I]Y(t) = 0$. Multiplying $[-W(t), I]$ from the left side of (1.2a) leads to $[-W(t), I]\dot{Y}(t) = [-W(t), I]BY(t)$. It follows that

$$\begin{aligned}\dot{W} &= \dot{Y}_2Y_1^{-1} + Y_2\frac{d}{dt}[Y_1^{-1}] = \dot{Y}_2Y_1^{-1} - Y_2Y_1^{-1}\dot{Y}_1Y_1^{-1} \\ &= [-W, I]\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} Y_1^{-1} = [-W, I]B\begin{bmatrix} I \\ W \end{bmatrix} \\ &= B_{21} - WB_{11} + B_{22}W - WB_{12}W,\end{aligned}$$

and $W(0) = Y_2(0)Y_1(0)^{-1}$. Hence, $W(t)$ for $t \in (a, b)$ is the solution of RDE (5.1).

Suppose that a, b are finite. To show that (a, b) is the maximal interval of existence of the solution of RDE (5.1), it suffices to prove that $W(t)$ blows-up as t approaches to a and b. Since $Y(t)$ for $t \in \mathbb{R}$ is orthogonal and (a, b) is the connected component of the open set $\{t \in \mathbb{R} \mid \det Y_1(t) \neq 0\}$, we see that $Y(b)$ is orthogonal and $Y_1(b)$ is singular. It turns out that there exists a nonzero vector $\mathbf{u} \in \mathbb{R}^r$ such that $Y_1(b)\mathbf{u} = \mathbf{0}$ and the vector $\mathbf{v} = Y_2(b)\mathbf{u}$ is a nonzero vector. By the continuity of $Y(t)$, we have

$$\lim_{t\to b^-} Y_1(t)\mathbf{u} = \mathbf{0} \text{ and } \lim_{t\to b^-} Y_2(t)\mathbf{u} = \mathbf{v}.$$

Let $\mathbf{y}(t) = Y_1(t)\mathbf{u}$. Then $\lim_{t\to b^-} \mathbf{y}(t) = \mathbf{0}$ and

$$\lim_{t\to b^-} W(t)\mathbf{y}(t) = \lim_{t\to b^-} Y_2(t)\mathbf{u} = \mathbf{v} \neq \mathbf{0}.$$

This implies that $W(t)$ blows-up as t approaches to b. The blowing-up of $W(t)$ at $t = a$ can be accordingly obtained. □

Remark 5.1. *Let*

$$X(t) \equiv \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = e^{Bt}Y_0,$$

be the solution of IVP (3.1)*. At* $t = 0$*,* $X(0) = Y_0$*. Consider the assumption of Theorem 5.1, that is,* $X_1(0) = Y_1(0)$ *is nonsingular. By the work of Randon, if* $W_0 = X_2(0)X_1^{-1}(0)$*, then* $W(t) = X_2(t)X_1^{-1}(t)$ *is a solution of RDE* (5.1) *whenever* $X_1(t)$ *is nonsingular. Therefore, we can see that Theorem 5.1 is an analogue of Radon's Lemma in [1].*

The next, we will study how a RDE transforms to a generalized orthogonal flow. Suppose that $W(t)$ for $t \in (a,b)$ is a solution of RDE (5.1). By Proposition 2.3, there exist an orthogonal matrix $Z(t) \in \mathbb{R}^{n\times r}$ and an upper triangular matrix $R_Z(t) \in \mathbb{R}^{r\times r}$ with a positive diagonal such that

$$\begin{bmatrix} I_r \\ W(t) \end{bmatrix} R_Z(t) = Z(t). \tag{5.2}$$

In addition, the matrix function $R_Z(t)$ is a $\mathcal{C}^1$ function on (a,b). Using the fact that $R_Z(t)$ is invertible and (5.2), we have

$$I + W(t)^\top W(t) = R_Z(t)^{-\top} R_Z(t)^{-1}. \tag{5.3}$$

Next, we show that the matrix function $Z(t)$ created by the solution $W(t)$ of RDE (5.1) is a generalized orthogonal flow, i.e., it is a solution of IVP (4.5) with $Z(0) = \widehat{Y}_0$.

Theorem 5.2. *Suppose that $W(t)$ for $t \in (a,b)$ is the solution of RDE* (5.1)*. Let $Z(t)$ in* (5.2) *be orthogonal. Then there is a continuous function $K(t)$ with $K(t)^\top = -K(t)$ defined on (a,b) such that $Z(t)$ satisfies*

$$\dot{Z} = BZ + Z(-\frac{1}{2}Z^\top(B^\top + B)Z + K), \quad Z(0) = \begin{bmatrix} I_r \\ W_0 \end{bmatrix} R_Z(0). \tag{5.4}$$

Proof. Suppose that $W(t)$ for $t \in (a,b)$ is the solution of RDE (5.1). By Proposition 2.3, the upper triangular matrix function $R_Z(t)$ in (5.2) is of class $\mathcal{C}^1$ on (a,b). Taking derivatives on both sides of (5.3), it follows from (5.1) that

$$\begin{aligned} -R_Z^{-\top}\dot{R}_Z^\top R_Z^{-\top}R_Z^{-1} - R_Z^{-\top}R_Z^{-1}\dot{R}_Z R_Z^{-1} &= \dot{W}^\top W + W^\top \dot{W} \\ &= [I, W^\top]B^\top \begin{bmatrix} -W^\top \\ I \end{bmatrix} W + W^\top[-W, I]B \begin{bmatrix} I \\ W \end{bmatrix}. \end{aligned}$$

This implies that

$$\dot{R}_Z^\top R_Z^{-\top} + R_Z^{-1}\dot{R}_Z = -Z^\top B^\top \begin{bmatrix} -W^\top \\ I \end{bmatrix} W R_Z - R_Z^\top W^\top[-W, I]BZ,$$

where $Z = Z(t)$ given in (5.2) is also a $\mathcal{C}^1$ matrix function on (a,b). Since R_Z is upper triangular, so is $R_Z^{-1}\dot{R}_Z$. From Proposition 2.2, we have

$$R_Z^{-1}\dot{R}_Z = -\frac{1}{2}\left(Z^\top B^\top \begin{bmatrix} -W^\top \\ I \end{bmatrix} W R_Z + R_Z^\top W^\top[-W, I]BZ\right) + S, \tag{5.5}$$

where $S = S(t)$ for $t \in (a,b)$ is skew symmetric such that the right hand side of (5.5) is upper triangular. Here, $S(t)$ is continuous on (a,b) because R_Z^{-1} and R_Z are of class $\mathcal{C}^1$. Using the equation (5.3), we obtain that

$$\begin{bmatrix} -W^\top \\ I \end{bmatrix} W = \begin{bmatrix} -W^\top W \\ W \end{bmatrix} = \begin{bmatrix} I - R_Z^{-\top} R_Z^{-1} \\ W \end{bmatrix} = \begin{bmatrix} I \\ W \end{bmatrix} - \begin{bmatrix} R_Z^{-\top} R_Z^{-1} \\ 0 \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} -W^\top \\ I \end{bmatrix} W R_Z = Z - \begin{bmatrix} R_Z^{-\top} \\ 0 \end{bmatrix}.$$

Hence, (5.5) can be rewritten as

$$\begin{aligned} R_Z^{-1}\dot{R}_Z &= -\frac{1}{2}Z^\top \left(B^\top + B\right) Z + \frac{1}{2}Z^\top B^\top \begin{bmatrix} R_Z^{-\top} \\ 0 \end{bmatrix} + \frac{1}{2}[R_Z^{-1}, 0]BZ + S \\ &= [R_Z^{-1}, 0]BZ - \frac{1}{2}Z^\top \left(B^\top + B\right) Z + \left(S + \frac{1}{2}Z^\top B^\top \begin{bmatrix} R_Z^{-\top} \\ 0 \end{bmatrix} - \frac{1}{2}[R_Z^{-1}, 0]BZ\right) \\ &= [R_Z^{-1}, 0]BZ - \frac{1}{2}Z^\top \left(B^\top + B\right) Z + K, \end{aligned} \tag{5.6}$$

where

$$K = S + \frac{1}{2}Z^\top B^\top \begin{bmatrix} R_Z^{-\top} \\ 0 \end{bmatrix} - \frac{1}{2}[R_Z^{-1}, 0]BZ \tag{5.7}$$

is skew-symmetric. Here $K = K(t)$ is a continuous matrix function defined on (a,b).

Taking derivatives on both sides of (5.2) and using equations (5.2) and (5.6), we have

$$\begin{aligned} \dot{Z} &= \frac{d}{dt}\left(\begin{bmatrix} I \\ W \end{bmatrix} R_Z\right) = \begin{bmatrix} 0 \\ \dot{W} \end{bmatrix} R_Z + \begin{bmatrix} I \\ W \end{bmatrix} \dot{R}_Z \\ &= \begin{bmatrix} 0 \\ \begin{bmatrix} -W, & I \end{bmatrix} BZ \end{bmatrix} + Z(R_Z^{-1}\dot{R}_Z) \\ &= \begin{bmatrix} 0 & 0 \\ -W & I \end{bmatrix} BZ + \begin{bmatrix} I & 0 \\ W & 0 \end{bmatrix} BZ + Z(-\frac{1}{2}Z^\top \left(B^\top + B\right) Z + K) \\ &= BZ + Z(-\frac{1}{2}Z^\top (B^\top + B)Z + K), \end{aligned}$$

where K defined in (5.7) is skew symmetric. This completes the proof. $\square$

Remark 5.2. *Suppose that* $W(t) \in \mathbb{R}^{(n-r)\times r}$ *for* $t \in (a,b)$ *is the solution of RDE* (5.1)*. The columns of* $Z(t) \in \mathbb{R}^{n\times r}$ *given in* (5.2) *for* $t \in (a,b)$ *form an orthonormal basis for column space of* $\begin{bmatrix} I_r \\ W(t) \end{bmatrix}$*. From Theorem 4.1* (*i*) *and Theorem 5.2,* $Z(t)$ *is a generalized orthogonal flow. Consequently, it follows from Theorem 4.1* (*ii*) *that there exists an orthogonal matrix flow* $Q(t) \in \mathbb{R}^{r\times r}$ *such that* $Z(t) = Y(t)Q(t)$*, where* $Y(t)$ *is the orthogonal flow which satisfies the IVP* (1.2) *with initial* $Y_0 = Z(0)$*. Hence, we have*

$$\begin{bmatrix} I_r \\ W(t) \end{bmatrix} R_Z(t) = Y(t)Q(t) \text{ for } t \in (a,b). \tag{5.8}$$

This implies that the solution of RDE can be transformed into a flow that can be represented by the orthogonal flow $Y(t)$ *multiplied by an orthogonal matrix* $Q(t)$ *from the right.*

We know that the solution of RDE (5.1) may blow-up at some finite time t. Extended solution of RDE can be obtained by using the Radon's Lemma and Grassmann manifolds approach [1]. In the following remark, we use the orthogonal flow to extend the domain of the solution of RDE.

Remark 5.3. *The solution of RDE* (5.1)*,* $W(t) \in \mathbb{R}^{(n-r)\times r}$ *for* $t \in (a,b)$*, may blow-up as* t *approaches* b*. Let* $Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$ *for* $t \in \mathbb{R}$ *be the solution of IVP* (1.2) *with* $Y_0 = Z(0)$*, where* $Z(0)$ *is given in* (5.2)*. From* (5.8)*, we have*

$$W(t) = Y_2(t)Y_1(t)^{-1} \text{ for } t \in (a,b).$$

We should note that $Y_1(b)$ *is singular, because* $W(t)$ *blows-up as* t *approaches* b*. Since the orthogonal flow* $Y(t)$ *exists for all* $t \in \mathbb{R}$*, let* $\mathcal{T}_W = \{t \in \mathbb{R} \mid \det Y_1(t) \neq 0\}$*, the extended solution of RDE* (5.1) *can be defined as*

$$W(t) = Y_2(t)Y_1(t)^{-1} \text{ for } t \in \mathcal{T}_W.$$

6 Concluding Remarks

In this paper, we study the fundamental properties of the orthogonal flow (1.2), such as the existence and uniqueness of the solution to (1.2) and the orthonormalization of its solution. Besides, we also show that the flow connects the sequence of matrices generated by the orthogonal iteration (1.1) in which $A = e^B$ and Y_0 are provided. Moreover, a generalized orthogonal flow is defined. We study the relationship between the orthogonal flow and the RDE. We have some remarks on future works.

1. Based on Remark 5.3, we see that the extended solution of RDE (5.4) is $W(t) = Y_2(t)Y_1^{-1}(t)$ where $Y(t)$ is the solution of IVP (1.2). From the viewpoint of computation, the orthogonal flow (1.2) can be used for computing the extended solution of a RDE numerically.

2. The case that the matrix A has Jordan blocks of eigenvalue zero causes that the matrix B does not exist. It turns out that the IVP (1.2) does no longer exist. This case is our next work in the future.

3. It is of our interest in the study of the existence and uniqueness for generalized orthogonal flow (4.5) with $K = K(\widehat{Y}, t)$.

References

[1] Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations: In Control and Systems Theory*, Birkhauser, Basel 2003.

[2] Z. Aganovic and Z. Gajic, *Optimal control of weakly coupled bilinear systems*, Automatica, **29** (1993). pp. 1591–1593.

[3] T. M. Chu, *The generalized Toda flow, the QR algorithm and the center manifold theory*, SIAM J. Alg. Discrete Meth., 5 (1984), pp. 187–201.

[4] T. M. Chu, *On the global convergence of the Toda lattice for real normal matrices and its applications to the eigenvalue problem*, SIAM J. Matrix Anal. Appl., 15 (1984), pp. 98–104.

[5] T. M. Chu, *Linear algebra algorithms as dynamical systems*, Acta Numerica, 17 (2008), pp. 1–86.

[6] T. M. Chu and K. L. Norris, *Isospectral Flows and Abstract Matrix Factorizations*, SIAM J. Num. Anal., **25(6)** (1988), 1387-1391.

[7] L. Dieci, M. Osborne, and R. Russell, *A Riccati transformation method for solving linear BVPs. I: Theoretical aspects*, SIAM J. Numer. Anal., 25 (1988), pp. 1055–1073.

[8] L. Dieci, M. Osborne, and R. Russell, *A Riccati transformation method for solving linear BVPs. II: Computational aspects*, SIAM J. Numer. Anal., 25 (1988), pp. 1074–1092.

[9] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4rd ed., Johns Hopkins University Press, Baltimore, 2013.

[10] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge University Press, 1985.

[11] Y. C. Kuo, W. W. Lin, and S. F. Shieh, *Structure- Preserving Flows of Symplectic Matrix Pairs*, SIAM J. Matrix. Anal. Appl., **37(3)** (2016), pp. 976–1001.

[12] Y. C. Kuo, W. W. Lin, and S. F. Shieh, *The asymptotic analysis of the structure-preserving doubling algorithms*, Linear Algebra Appl. **531** (2017), pp. 318–355.

[13] Y. C. Kuo, W. W. Lin, and S. F. Shieh, *A structure preserving flow for computing Hamiltonian matrix exponential*, Numer. Math., **143** (2019), pp. 555–582.

[14] P. V. Kokotovic, R. E. O'Malley Jr, and P. Sannuti, *Singular perturbations and order reduction in control theory–an overview*, Automatica, **12** (1976). pp. 123–321.

[15] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

[16] A. J. Laub, *Schur techniques for Riccati differential equations*, in Feedback Control of Linear and Nonlinear Systems, (1982), pp. 165–174.

[17] W. W. Symes, *The QR algorithm and scattering for the finite nonperiodic Toda lattice*, Physica D: Nonlinear Phenomena, 4 (1982), pp. 275–280.

108年度專題研究計畫成果彙整表

計畫主持人：謝世峰			計畫編號：108-2115-M-003-013-		
計畫名稱：黎卡提微分方程之混沌動態及矩陣指數上的應用					
成果項目			量化	單位	質化 （說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文	1	篇	Taiwanese J. Math. Volume 24, Number 1 (2020), 131-158. Time-asymptotic Dynamics of Hermitian Riccati Differential Equations Yueh-Cheng Kuo, Huey-Er Lin, and Shih-Feng Shieh
		研討會論文	0		
		專書	0	本	
		專書論文	0	章	
		技術報告	0	篇	
		其他	0	篇	
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		專書	0	本	
		專書論文	0	章	
		技術報告	0	篇	
		其他	0	篇	
參與計畫人力	本國籍	大專生	0	人次	
		碩士生	0		
		博士生	0		
		博士級研究人員	0		
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其他成果 （無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）					